Rational Inference Relations from Maximal Consistent Subsets Selection

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Abstract

When one wants to draw non-trivial inferences from an inconsistent belief base, a very natural approach is to take advantage of the maximal consistent subsets of the base. But few inference relations from maximal consistent subsets exist. In this paper we point out new inference relations based on selection of some maximal consistent subsets, leading thus to inference relations with a stronger inferential power. The selection process must obey some principles to ensure that it leads to an inference relation which is rational. We define a general class of monotonic selection relations for comparing maximal consistent subsets and show that it corresponds to the class of rational inference relations.

1 Introduction

A very natural approach to draw plausible conclusions from an inconsistent belief base amounts to taking advantage of the maximal (for set inclusion) consistent subsets of the base (see e.g., [Rescher and Manor, 1970]). Quite surprisingly, very few inference mechanisms exist for defining an inference relation that is based on those subsets. Indeed, to the best of our knowledge, only three mechanisms have been identified so far in the literature (see e.g., [Pinkas and Loui, 1991; Benferhat *et al.*, 1997]:

- skeptical inference, where a formula is a consequence of the base if it is implied by each maximal consistent subset,
- credulous inference, where a formula is a consequence of the base if it is implied by at least one maximal consistent subset,
- argumentative inference, where a formula is a consequence of the base if it is implied by at least one maximal consistent subset and its negation is not.

However, the credulous and the argumentative mechanisms are not satisfactory since they do not induce so-called preferential inference relations, i.e., they miss to satisfy some expected postulates for inference relations [Kraus *et al.*, 1990]. Worse than that, credulous inference does not lead to jointly consistent conclusions since for inconsistent bases, it may be possible to infer credulously both a formula and its negation. Accordingly, the only fully rational choice identified in the literature is skeptical inference: it is the only inference mechanism that leads to preferential inference relations [Kraus *et al.*, 1990]. The basic inference relation based on the skeptical inference mechanism considers every maximal consistent subset of the base. A well-known refinement of this relation consists in focusing on the largest maximal consistent subsets of the base. This relation is known to be rational [Benferhat *et al.*, 1993].

The main question we want to address in this paper is to determine whether other refinements of the basic inference relation satisfying the expected postulates for nonmonotonic inference exist, and if the answer is positive, to characterize some of them. This is important because the inferential power of the inference relation based on all (or only the largest) maximal consistent subsets is sometimes very weak, as illustrated by the following example.

Consider a scenario where a belief base is made by putting together in a common repository some pieces of information, issued from several sources which are equally reliable, and represented by propositional formulae. Suppose also that the origins of the pieces of information (e.g., the sources they come from) are unknown or have been lost. This is a common assumption underlying for instance the AGM setting for belief revision, where it is supposed as well that one cannot trace back the pieces of beliefs [Gärdenfors, 1988; Gärdenfors, 1992]. In such a case, one cannot take advantage of them to make a selection of the subsets, as done for instance in belief merging [Konieczny, 2000; Konieczny and Pérez, 2002; Konieczny and Pérez, 2011]. For the sake of illustration, suppose that the (contradictory) pieces of information that were gathered concern an incoming model of car from our favorite brand (those pieces of information have been obtained through different sources, like car magazines, websites, friends, etc.). The first piece of information is that the new car has a 6-cylinder engine: $\varphi_1 = e6c$. The second piece of information is that it has a manual gearbox: $\varphi_2 = mg$. The third one is that it has a turbo (t) and that a car that has a 6-cylinder engine is a not a sport car: $\varphi_3 = t \wedge (e6c \rightarrow \neg sc)$. The fourth one is that the car has a low fuel consumption (lc) and does not have a manual gearbox: $\varphi_4 = lc \wedge \neg mg$. The fifth one is that it is a sport car (sc), that does not have a manual gearbox and that a sport car does not have a low fuel consumption, and that a car with a 6-cylinder engine does not have 4-wheel drive $(wd4)$: $\varphi_5 = sc \wedge \neg mg \wedge (sc \rightarrow \neg lc) \wedge (e6c \rightarrow \neg wd4)$. The last one is that the car has 4-wheel drive, does not have a manual gearbox, and that cars with 4-wheel drive do not have a low fuel consumption, and that cars with 6-cylinder engine do not need a turbo: $\varphi_6 = wd4 \wedge \neg mg \wedge (wd4 \rightarrow \neg lc) \wedge (e6c \rightarrow \neg t)$.

To sum up, the belief base is $\mathtt{K} = \{\varphi_1, \ldots, \varphi_6\}$ where:

$\varphi_1 = e6c$ $\qquad\qquad \varphi_2 = mg$

$\varphi_3 = t \wedge (e6c \rightarrow \neg sc)$ $\qquad \varphi_4 = lc \wedge \neg mg$

$\varphi_5 = sc \wedge \neg mg \wedge (sc \rightarrow \neg lc) \wedge (e6c \rightarrow \neg wd4)$

$\varphi_6 = wd4 \wedge \neg mg \wedge (wd4 \rightarrow \neg lc) \wedge (e6c \rightarrow \neg t)$

It can be easily checked that $\mathtt{K}$ has 5 maximal consistent subsets, namely: $\mathtt{K}_1 = \{\varphi_1, \varphi_6\}$, $\mathtt{K}_2 = \{\varphi_1, \varphi_5\}$, $\mathtt{K}_3 = \{\varphi_1, \varphi_2, \varphi_3\}$, $\mathtt{K}_4 = \{\varphi_1, \varphi_3, \varphi_4\}$, $\mathtt{K}_5 = \{\varphi_3, \varphi_5, \varphi_6\}$.

Using skeptical inference from all these maximal consistent subsets, none of the six formulae in $\mathtt{K}$ can be derived as a conclusion. This means that whenever the formula φ_i (with $i \in \{1, \ldots, 6\}$), there exists a maximal consistent subset $\mathtt{K}_j$ ($j \in \{1, \ldots, 5\}$) of $\mathtt{K}$ such that $\mathtt{K}_j \models \neg\varphi_i$. Similarly, only φ_3 can be derived from $\mathtt{K}$ provided that the largest maximal consistent subsets of $\mathtt{K}$ (here, $\mathtt{K}_3$, $\mathtt{K}_4$, and $\mathtt{K}_5$) are considered. As a consequence, only very weak conclusions composed of disjunctions of those formulae can be obtained as consequences. However, all the incoming pieces of information φ_i do not play symmetric role with respect to the maximal consistent subsets $\mathtt{K}_j$. Consider for instance φ_1 and φ_2: we have that φ_1 is a logical consequence of 4 (over 5) maximal consistent subsets of K, while φ_2 is a logical consequence of only one of them. Since the global inconsistency of a belief base is often due to the presence in it of erroneous pieces of information, it makes sense to take advantage of this discrepancy to consider some pieces of information as more reliable than others because they are more consensual / less conflicting with the other pieces of information which have been reported.

In the following a new family of inference relations from maximal consistent subsets is presented. The key idea underlying them is to select some of the maximal consistent subsets of the base. This allows us to define inference relations with a stronger inferential power than the standard inference relation focusing on the consequences of all maximal consistent subsets. However, making the selection process arbitrary does not guarantee that the resulting inference relation is preferential, which is expected. Some principles are needed. In this direction, we define a general class of monotonic selection relations for comparing maximal consistent subsets. We provide several examples of inference relations from this class. We show that, whenever a monotonic selection relation is used to select the best maximal consistent subsets, the induced inference relation is preferential. Furthermore, it also satisfies rational monotony, i.e., it is a rational inference relation in the sense of [Lehmann and Magidor, 1992]. More than that, we provide a representation theorem showing that the class of rational inference relations of [Lehmann and Magidor, 1992] coincides with the class of skeptical inference relations from selected maximal consistent subsets, where the selection process is achieved using a monotonic selection relation.

2 Preliminaries

Our formal setting is classical propositional logic. Thus we consider a language $\mathcal{L}$ defined from a finite set $\mathcal{V}$ of propositional variables and the usual connectives. The elements of $\mathcal{L}$ are called *formulae*. An interpretation m is a mapping that assigns a truth value to every variable of $\mathcal{V}$. We denote the set of interpretations of propositional logic by $\mathtt{Mod}(\mathcal{L})$. For a formula α and an interpretation $m \in \mathtt{Mod}(\mathcal{L})$, m is a model of α, denoted $m \models \alpha$ if α is true in m. For a formula α, we denote by $\mathtt{Mod}(\alpha)$ the set of its models. We say that a model m is a model of a set of formulae S and write $m \models S$ if for every formula $\alpha \in S$, we have $m \models \alpha$. For two formulae α, β, we write $\alpha \models \beta$ if every model of α is a model of β. For a set of formulae S and a formula α, we write $S \models \alpha$ if every model of S is a model of α. Given a set of formulae S, we denote $\mathtt{Cn}(S) = \{\alpha \in \mathcal{L} \mid S \models \alpha\}$, the set of consequences of S. We say that a set S is consistent if and only if it has at least one model. We say that a formula α is consistent if and only if the set $\{\alpha\}$ is consistent, and that α is a trivial formula if and only if $\alpha \equiv \top$ or $\alpha \equiv \bot$.

Definition 1 (Belief base)**.** *A* belief base *is a finite set of formulae.*

Standard notions when facing inconsistent belief bases are minimal inconsistent subsets, that encode the sources of conflicts in the base; and maximal consistent subsets, which can be considered as the potential repairs of the inconsistent belief base.

Definition 2 ($\mathtt{mc}$)**.** $\mathtt{mc}$ *is a mapping defined as follows: for every belief base* $\mathtt{K}$*,* $\mathtt{mc}(\mathtt{K})$ *is the set of all maximal (for set inclusion) consistent subsets of* $\mathtt{K}$*:*

- $\mathtt{K}' \subseteq \mathtt{K}$
- $\mathtt{K}'$ *is consistent*
- *If* $\mathtt{K}' \subset \mathtt{K}'' \subseteq \mathtt{K}$*, then* $\mathtt{K}''$ *is not consistent*

Definition 3 ($\mathtt{mus}$)**.** $\mathtt{mus}$ *is a mapping defined as follows: for every belief base* $\mathtt{K}$*,* $\mathtt{mus}(\mathtt{K})$ *is the set of all minimal (for set inclusion) inconsistent subsets of* $\mathtt{K}$*:*

- $\mathtt{K}' \subseteq \mathtt{K}$
- $\mathtt{K}'$ *is not consistent*
- *If* $\mathtt{K}'' \subset \mathtt{K}'$*, then* $\mathtt{K}''$ *is consistent*

3 Inference from Selected Maximal Consistent Subsets

Our main objective is to identify selection criteria on maximal consistent subsets, leading to minimize information loss from the belief base while guaranteeing that the induced inference relations are rational ones in the sense of [Lehmann and Magidor, 1992]. One already knows that such criteria exist, since the inference relation based on the largest maximal consistent subsets of $\mathtt{K}$ is rational [Benferhat *et al.*, 1993], but we would like to identify other inference schemata and more general conditions on the set of all maximal consistent subsets which are sufficient to ensure that the corresponding inference relations are rational. To this end, we introduce new inference relations based on a selection of the maximal consistent subsets of $\mathtt{K}$ maximizing a given scoring function.

We will first define mappings that attach a score to each formula α of $\mathtt{K}$ and then aggregate those scores (in the following, we will simply sum up the scores; as will be discussed later, other aggregation functions could be considered alternatively).

Definition 4 (scoring function)**.** *A* scoring function *s associates with a belief base $\mathtt{K}$ and a formula $\alpha \in \mathtt{K}$ a non-negative real number $s(\mathtt{K}, \alpha)$ which is equal to 0 if and only if α is a trivial formula (i.e., such that $\alpha \equiv \top$ or $\alpha \equiv \bot$).*

Here are some examples of scoring functions. The first one is based on the number of maximal consistent sets a formula belongs to.

Definition 5 ($\#\mathtt{mc}$)**.** *Let $\mathtt{K}$ be a belief base and $\alpha \in \mathtt{K}$. We define:*

$$\#\mathtt{mc}(\mathtt{K}, \alpha) = \begin{cases} 0 & \textit{if } \alpha \textit{ is trivial} \\ |\{\mathtt{K}_i \in \mathtt{mc}(\mathtt{K}) \mid \alpha \in \mathtt{K}_i\}| & \textit{otherwise} \end{cases}$$

Another interesting example of a scoring function is based on the number of minimal inconsistent sets a formula belongs to. The scale must be reversed here, since this number must be minimized if one wants to give some preference to the less conflicting formulae. The addition of 1 in the second part of the definition is introduced in order to make sure that only trivial formulae can get the score 0.

Definition 6 ($\#\mathtt{mus}$)**.** *Let $\mathtt{K}$ be a belief base and $\alpha \in \mathtt{K}$. We define:*

$$\#\mathtt{mus}(\mathtt{K}, \alpha) = \begin{cases} 0 \textit{ if } \alpha \textit{ is trivial, otherwise} \\ 1 + |\mathtt{mus}(\mathtt{K})| - \\ \qquad |\{\mathtt{K}_i \subseteq \mathtt{K} \mid \mathtt{K}_i \in \mathtt{mus}(\mathtt{K}), \alpha \in \mathtt{K}_i\}| \end{cases}$$

We could also use an inconsistency measure in order to attach scores to formulae. Let us consider the measure MIV, introduced by Hunter and Konieczny (2006; 2010), which is based both on the number of minimal inconsistent subsets containing a formula and on their cardinalities. The idea is that belonging to a large inconsistent set puts less blame on a formula than belonging to a small set.

Definition 7 ($\mathtt{MIV}$)**.** *Let $\mathtt{K}$ be a belief base and $\alpha \in \mathtt{K}$. We define:*

$$\mathtt{MIV}_{\mathtt{K}}(\alpha) = \sum_{M \in \mathtt{mus}(\mathtt{K}), \alpha \in M} \frac{1}{|M|}$$

We can use $\mathtt{MIV}$ to define a scoring function $\mathtt{miv}$ as follows:

Definition 8 ($\mathtt{miv}$)**.** *Let $\mathtt{K}$ be a belief base and $\alpha \in \mathtt{K}$. We define $\mathtt{maxmiv}(\mathtt{K}) = max_{\alpha \in \mathtt{K}} \mathtt{MIV}_{\mathtt{K}}(\alpha)$ and*

$$\mathtt{miv}(\mathtt{K}, \alpha) = \begin{cases} 0 & \textit{if } \alpha \textit{ trivial} \\ 1 + \mathtt{maxmiv}(\mathtt{K}) - \mathtt{MIV}_{\mathtt{K}}(\alpha) & \textit{otherwise} \end{cases}$$

On this ground, the score of any subset of $\mathtt{K}$ can be computed by aggregating (e.g., using sum) the scores of its elements:

Definition 9 ($\mathtt{score}^s_{\mathtt{K},sum}$)**.** *Let s be a scoring function. Let $\mathtt{K}$ be a belief base, and let $\mathtt{K}_i \subseteq \mathtt{K}$. We note $\mathtt{score}^s_{\mathtt{K},sum}(\mathtt{K}_i) = \sum_{\alpha \in \mathtt{K}_i} s(\mathtt{K}, \alpha)$.*

Especially, we have:

$$\mathtt{score}^{\#mc}_{\mathtt{K},sum}(\mathtt{K}_i) = \sum_{\alpha \in \mathtt{K}_i} \#\mathtt{mc}(\mathtt{K}, \alpha).$$

$$\mathtt{score}^{\#mus}_{\mathtt{K},sum}(\mathtt{K}_i) = \sum_{\alpha \in \mathtt{K}_i} \#\mathtt{mus}(\mathtt{K}, \alpha).$$

$$\mathtt{score}^{\mathtt{miv}}_{\mathtt{K},sum}(\mathtt{K}_i) = \sum_{\alpha \in \mathtt{K}_i} \mathtt{miv}(\mathtt{K}, \alpha).$$

The reader may observe that $\mathtt{score}^{\#mc}_{\mathtt{K},sum}$ was the scoring function which was informally used in the introduction.

Let us now show how to infer conclusions from a belief base. We first need the following notation that will prove convenient: $\mathtt{mc}(\mathtt{K}, \alpha) = \{\mathtt{K}_i \subseteq \mathtt{K} \mid \mathtt{K}_i \cup \{\alpha\} \in \mathtt{mc}(\mathtt{K} \cup \{\alpha\})\}$.

Definition 10 (Inference from subsets with best scores)**.** *Let $\mathtt{K}$ be a belief base and α and β be two formulae. Let s be a scoring function. We define $\mathtt{mc}_{\mathtt{score}^s_{\mathtt{K},sum}}(\mathtt{K}, \alpha) = \{\mathtt{K}_i \in \mathtt{mc}(\mathtt{K}, \alpha)$ and there exists no $\mathtt{K}'_i \in \mathtt{mc}(\mathtt{K}, \alpha)$ such that $\mathtt{score}^s_{\mathtt{K},sum}(\mathtt{K}'_i) > \mathtt{score}^s_{\mathtt{K},sum}(\mathtt{K}_i)\}$. We say that $\alpha \mid\!\sim^s_{\mathtt{K},sum} \beta$ if and only if either α is inconsistent, or for every $\mathtt{K}_i \in \mathtt{mc}_{\mathtt{score}^s_{\mathtt{K},sum}}(\mathtt{K}, \alpha)$ we have $\mathtt{K}_i \cup \{\alpha\} \models \beta$.*

4 Examples

Let us now illustrate the introduced notions by stepping back to sport car example given in the introduction.

Example 1. *We have $\#\mathtt{mc}(\mathtt{K}, \varphi_1) = 4$, $\#\mathtt{mc}(\mathtt{K}, \varphi_2) = 1$, $\#\mathtt{mc}(\mathtt{K}, \varphi_3) = 3$, $\#\mathtt{mc}(\mathtt{K}, \varphi_4) = 1$, $\#\mathtt{mc}(\mathtt{K}, \varphi_5) = 2$, $\#\mathtt{mc}(\mathtt{K}, \varphi_6) = 2$, so the score of K_1 (resp. K_2, K_3, K_4, K_5) w.r.t. $\mathtt{score}^{\#\mathtt{mc}}_{\mathtt{K},sum}$ is equal to 6 (resp. 6, 8, 8, 7). Hence, the best sets are $\mathtt{K}_3$ and $\mathtt{K}_4$. Therefore, we have that*

$$\top \mid\!\sim^{\#\mathtt{mc}}_{\mathtt{K},sum} e6c \wedge t \wedge \neg sc.$$

Hence, we conclude that the car has a 6-cylinder engine, a turbo, and is not a sport car.

We obtain exactly the same result with $\#\mathtt{mus}$ and $\mathtt{miv}$, so we do not give the details of the computation here. But let us illustrate on the next example that the three inference relations based respectively on the scoring functions $\#\mathtt{mc}$, $\#\mathtt{mus}$, and $\mathtt{miv}$ leads to distinct sets of consequences in the general case.

Example 2. *Let $\mathtt{K} = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7\}$ with $\varphi_1 = a \wedge b$, $\varphi_2 = a \wedge (c \vee d)$, $\varphi_3 = a \wedge \neg d$, $\varphi_4 = a \wedge \neg c \wedge e$, $\varphi_5 = \neg a \wedge \neg b$, $\varphi_6 = a \wedge (\neg c \rightarrow \neg e)$, $\varphi_7 = a \wedge \neg c \wedge f$*

$\mathtt{K}$ has 9 minimal unsatisfiable subsets: $M_1 = \{\varphi_1, \varphi_5\}$, $M_2 = \{\varphi_2, \varphi_5\}$, $M_3 = \{\varphi_3, \varphi_5\}$, $M_4 = \{\varphi_4, \varphi_5\}$, $M_5 = \{\varphi_5, \varphi_6\}$, $M_6 = \{\varphi_5, \varphi_7\}$, $M_7 = \{\varphi_4, \varphi_6\}$, $M_8 = \{\varphi_2, \varphi_3, \varphi_7\}$, $M_9 = \{\varphi_2, \varphi_3, \varphi_4\}$.

$\mathtt{K}$ has five maximal consistent subsets: $\mathtt{K}_1 = \{\varphi_1, \varphi_2, \varphi_4, \varphi_7\}$, $\mathtt{K}_2 = \{\varphi_1, \varphi_2, \varphi_3, \varphi_6\}$, $\mathtt{K}_3 = \{\varphi_1, \varphi_3, \varphi_4, \varphi_7\}$, $\mathtt{K}_4 = \{\varphi_1, \varphi_3, \varphi_6, \varphi_7\}$, $\mathtt{K}_5 = \{\varphi_5\}$.

We have: $\#\mathtt{mc}(\mathtt{K}, \varphi_1) = 4$, $\#\mathtt{mc}(\mathtt{K}, \varphi_2) = 2$, $\#\mathtt{mc}(\mathtt{K}, \varphi_3) = 3$, $\#\mathtt{mc}(\mathtt{K}, \varphi_4) = 2$, $\#\mathtt{mc}(\mathtt{K}, \varphi_5) = 1$, $\#\mathtt{mc}(\mathtt{K}, \varphi_6) = 2$, $\#\mathtt{mc}(\mathtt{K}, \varphi_7) = 3$, so that the score of K_1 (resp. K_2, K_3, K_4, K_5) w.r.t. $\mathtt{score}^{\#\mathtt{mc}}_{\mathtt{K},sum}$ is equal to 11

(resp. 11, 12, 12, 1*). Hence, the best sets are* $\mathtt{K}_3$ *and* $\mathtt{K}_4$*. Therefore, we have that* $\top \mathrel{|\!\sim}^{\#\mathtt{mc}}_{\mathtt{K},sum} a \wedge b \wedge \neg c \wedge \neg d \wedge f$.

Let us count the number of minimal unsatisfiable sets containing a given formula; we have $\#\mathtt{mus}(\mathtt{K}, \varphi_1) = 6$, $\#\mathtt{mus}(\mathtt{K}, \varphi_2) = 4$, $\#\mathtt{mus}(\mathtt{K}, \varphi_3) = 4$, $\#\mathtt{mus}(\mathtt{K}, \varphi_4) = 4$, $\#\mathtt{mus}(\mathtt{K}, \varphi_5) = 1$, $\#\mathtt{mus}(\mathtt{K}, \varphi_6) = 5$, $\#\mathtt{mus}(\mathtt{K}, \varphi_7) = 5$*, so the score of* K_1 *(resp.* K_2, K_3, K_4, K_5*) w.r.t.* $\mathtt{score}^{\#\mathtt{mus}}_{\mathtt{K},sum}$ *is equal to* 15 *(resp.* 19, 19, 20, 1*). Hence, the best set is* $\mathtt{K}_4$*. And we have* $\top \mathrel{|\!\sim}^{\#\mathtt{mus}}_{\mathtt{K},sum} a \wedge b \wedge \neg c \wedge \neg d \wedge \neg e \wedge f$*. Note that we can also conclude* $\neg e$*, which was not the case using* $\#\mathtt{mc}$.

Let us now show how to compute scores using a Shapley inconsistency value. We have (rounded on two decimals): $\mathtt{miv}(\mathtt{K}, \varphi_1) = 3.50$, $\mathtt{miv}(\mathtt{K}, \varphi_2) = 2.83$, $\mathtt{miv}(\mathtt{K}, \varphi_3) = 2.83$, $\mathtt{miv}(\mathtt{K}, \varphi_4) = 3.00$, $\mathtt{miv}(\mathtt{K}, \varphi_5) = 1.00$, $\mathtt{miv}(\mathtt{K}, \varphi_6) = 3.00$, $\mathtt{miv}(\mathtt{K}, \varphi_7) = 3.17$*, so the score of* K_1 *(resp.* K_2, K_3, K_4, K_5*) w.r.t.* $\mathtt{score}^{\mathtt{miv}}_{\mathtt{K},sum}$ *is equal to* 12.50 *(resp.* 12.16, 12.50, 12.50, 1.00*). Hence, three sets are selected:* $\mathtt{K}_1$, $\mathtt{K}_3$, $\mathtt{K}_4$*. So we have* $\top \mathrel{|\!\sim}^{\mathtt{miv}}_{\mathtt{K},sum} a \wedge b \wedge \neg c \wedge f$*. This time, we are neither able to conclude* $\neg d$ *nor* $\neg e$*.*

5 Logical Properties

Let us now formally evaluate the introduced methods w.r.t. their logical rationality. There has been much work on the issue of determining the minimum logical properties that any nonmonotonic inference relation should satisfy [Gabbay, 1985; Makinson, 1994; Kraus *et al.*, 1990; Lehmann and Magidor, 1992]. There is now a wide consensus on the fact that the minimal set of expected properties is the one of *preferential inference relations* [Lehmann and Magidor, 1992] (also called *system P*), and that an interesting subclass is the one of *rational inference relations* [Kraus *et al.*, 1990] (also called *system R*).

Preferential inference relations are characterized by the following postulates:

Ref $\alpha \mathrel{|\!\sim} \alpha$

LLE $\dfrac{\models \alpha \leftrightarrow \beta, \ \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma}$

RW $\dfrac{\models \alpha \rightarrow \beta, \ \gamma \mathrel{|\!\sim} \alpha}{\gamma \mathrel{|\!\sim} \beta}$

Cut $\dfrac{\alpha \wedge \beta \mathrel{|\!\sim} \gamma, \ \alpha \mathrel{|\!\sim} \beta}{\alpha \mathrel{|\!\sim} \gamma}$

Or $\dfrac{\alpha \mathrel{|\!\sim} \gamma, \ \beta \mathrel{|\!\sim} \gamma}{\alpha \vee \beta \mathrel{|\!\sim} \gamma}$

CM $\dfrac{\alpha \mathrel{|\!\sim} \beta, \ \alpha \mathrel{|\!\sim} \gamma}{\alpha \wedge \beta \mathrel{|\!\sim} \gamma}$

A *rational inference relation* is a preferential relation that also satisfies the RM (*rational monotony*) postulate:

RM $\dfrac{\alpha \mathrel{|\!\not\sim} \neg \beta \quad \alpha \mathrel{|\!\sim} \gamma}{\alpha \wedge \beta \mathrel{|\!\sim} \gamma}$

In this section, we show that the three inference relations defined above, as well as many other from the same class that we will formally define soon, satisfy the above postulates.

We start by generalizing the choice of the aggregation function. Previously, we considered *sum* in order to aggregate the scores of the formulae, but many other aggregation functions can be used instead.

Definition 11 (Aggregation function)**.** $\oplus$ *is an* aggregation function *if for every positive integer* n*, for every non-negative real number* $x_1, \ldots, x_n$*,* $\oplus(x_1, \ldots, x_n)$ *is a non-negative real number.*

Definition 12 (Properties of aggregation function)**.** *An aggregation function* $\oplus$ *satisfies:*

- **Composition** *if* $\oplus(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ *implies* $\oplus(x_1, \ldots, x_n, z) \leq (y_1, \ldots, y_n, z)$
- **Decomposition** *if* $\oplus(x_1, \ldots, x_n, z) \leq (y_1, \ldots, y_n, z)$ *implies* $\oplus(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$
- **Symmetry** *if for every permutation* σ*,* $\oplus(x_1, \ldots, x_n) = \oplus(\sigma(x_1, \ldots, x_n))$
- **Monotonicity** *if for every* $z > 0$ *we have* $\oplus(x_1, \ldots, x_n, z) > \oplus(x_1, \ldots, x_n)$

Composition, Decomposition and Symmetry were introduced in [Konieczny *et al.*, 2004]. We add here a new property, Monotonicity, and slightly change Composition and Decomposition for dealing with tuples of different sizes.

Observe that *sum* satisfies the above conditions. Another well-known aggregation function satisfying the previous four properties is lexicographic aggregation [Konieczny *et al.*, 2004; Dubois *et al.*, 1996; Moulin, 1988].

One can then define the score of any subset of the input belief base:

Definition 13 ($\mathtt{score}^s_{\mathtt{K},\oplus}$)**.** *Let* s *be a scoring function and* $\oplus$ *an aggregation function. Let* $\mathtt{K}$ *be a belief base and* $\mathtt{K}_i \subseteq \mathtt{K}$ *with* $\mathtt{K}_i = \{\alpha_1, \ldots, \alpha_n\}$*.We define*

$$\mathtt{score}^s_{\mathtt{K},\oplus}(\mathtt{K}_i) = \oplus_{\alpha \in \mathtt{K}_i} s(\mathtt{K}, \alpha).$$

Based on those scores, one can compare the subsets of the belief base:

Definition 14 ($\succeq_{\mathtt{K},\oplus}$)**.** *Let* s *be a scoring function and* $\oplus$ *an aggregation function. Let* $\mathtt{K}$ *be a belief base,* $\mathtt{K}_i, \mathtt{K}_j \subseteq \mathtt{K}$*. We state that* $\mathtt{K}_i \succeq^s_{\mathtt{K},\oplus} \mathtt{K}_j$ *if and only if* $\mathtt{score}^s_{\mathtt{K},\oplus}(\mathtt{K}_i) \geq \mathtt{score}^s_{\mathtt{K},\oplus}(\mathtt{K}_j)$*.*

We will show that using a scoring function and an aggregation operator satisfying Composition, Decomposition, Symmetry and Monotonicity ensures to get a rational inference relation. However, the result we obtained is in fact much more general: it shows that selecting maximal consistent subsets of the belief base using a *monotonic selection relation* is enough for giving rise to a rational inference relation, whatever this selection relation is induced or not from a scoring function. Formally:

Definition 15 (Monotonic selection relation)**.** *Given a belief base* $\mathtt{K}$*, let* $\succeq_{\mathtt{K}} \subseteq 2^{\mathtt{K}} \times 2^{\mathtt{K}}$ *be a reflexive, transitive and total relation*[1] *over the powerset of* $\mathtt{K}$*.* $\succeq_{\mathtt{K}}$ *is said to be a* monotonic selection relation *if for every consistent set* $\mathtt{K}_i \subseteq \mathtt{K}$*, for every non-trivial formula* $\alpha \in \mathtt{K} \setminus \mathtt{K}_i$*,* $\mathtt{K}_i \cup \{\alpha\} \succ_{\mathtt{K}} \mathtt{K}_i$*.*

For instance, consider the relation $\succeq_{card}$ that compares the subsets of $\mathtt{K}$ based on their cardinality:

Definition 16 ($\succeq_{card}$)**.** *For every two subsets* $\mathtt{K}_i$*,* $\mathtt{K}_j$ *of* $\mathtt{K}$*,* $\mathtt{K}_i \succeq_{card} \mathtt{K}_j$ *if and only if* $|\mathtt{K}_i| \geq |\mathtt{K}_j|$*.*

Clearly enough, $\succeq_{card}$ satisfies the conditions from Definition 15, thus it is a monotonic selection relation.

On this ground, one can define a selection mechanism which consists in keeping only the best sets with respect to a monotonic selection relation:

[1] We use the standard notation $\mathtt{K}_i \succ_{\mathtt{K}} \mathtt{K}_j$ for $\mathtt{K}_i \succeq_{\mathtt{K}} \mathtt{K}_j$ and not($\mathtt{K}_j \succeq_{\mathtt{K}} \mathtt{K}_i$).

Definition 17 ($\mathtt{mc}_{\succeq_{\mathtt{K}}}$). *Given a belief base* $\mathtt{K}$*, a formula* $\alpha \in \mathcal{L}$*, and a monotonic selection selection relation* $\succeq_{\mathtt{K}}$*, we define* $\mathtt{mc}_{\succeq_{\mathtt{K}}}(K,\alpha) = \{\mathtt{K}_i \in \mathtt{mc}(\mathtt{K},\alpha) \mid$ *there exists no* $\mathtt{K}'_i \in \mathtt{mc}(\mathtt{K},\alpha)$ *such that* $\mathtt{K}'_i \succ_{\mathtt{K}} \mathtt{K}_i\}$.

We now show that all the scoring functions studied in this paper induce monotonic selection relations. Formally, we identify sufficient conditions under which a relation comparing the maximal consistent subsets based on a scoring function is a monotonic selection relation.

Proposition 1. *Let* $\mathtt{K}$ *be a belief base,* $\oplus$ *be an aggregation function satisfying Composition, Decomposition, Symmetry and Monotonicity and let* s *be any scoring function.* $\succeq^s_{\mathtt{K},\oplus}$ *is a monotonic selection relation.*

Proof. Let $\mathtt{K}_i, \mathtt{K}_j \subseteq \mathtt{K}$. Note that $\mathtt{score}^s_{\mathtt{K},\oplus}(\mathtt{K}_i)$, $\mathtt{score}^s_{\mathtt{K},\oplus}(\mathtt{K}_j)$ are non-negative real numbers. Since $\mathtt{K}_i \succeq^s_{\mathtt{K},\oplus} \mathtt{K}_j$ if and only if $\mathtt{score}^s_{\mathtt{K},\oplus}(\mathtt{K}_i) \geq \mathtt{score}^s_{\mathtt{K},\oplus}(\mathtt{K}_j)$ we conclude that $\succeq^s_{\mathtt{K},\oplus}$ is reflexive, transitive and total.

Let $\alpha \in \mathtt{K} \setminus \mathtt{K}_i$ be a non-trivial formula. Since $\oplus$ satisfies monotonicity, $\mathtt{K}_i \cup \{\alpha\} \succ^s_{\mathtt{K},\oplus} \mathtt{K}_i$. Thus, $\succeq^s_{\mathtt{K},\oplus}$ satisfies monotonicity. □

Let us now generalize Definition 10:

Definition 18 (Inference from best subsets wrt $\succeq_{\mathtt{K}}$). *Given a belief base* $\mathtt{K}$*, two formulae* α *and* β*, and a monotonic selection relation* $\succeq_{\mathtt{K}}$*, we state that* $\alpha \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \beta$ *if and only if either* α *is inconsistent, or for every* $\mathtt{K}_i \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(K,\alpha)$ *we have* $\mathtt{K}_i \cup \{\alpha\} \models \beta$.

We can now present one of the main results of the paper:

Proposition 2. *If* $\succeq_{\mathtt{K}}$ *is a monotonic selection relation, then* $\mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}}$ *is rational.*

Proof. **Ref.** If α is inconsistent, then the proof is trivial. So let α be consistent. Then $\mathtt{mc}(\mathtt{K},\alpha) \neq \emptyset$. Consequently, $\mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha) \neq \emptyset$. Trivially, for every $\mathtt{K}_i \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$, $\mathtt{K}_i \cup \alpha \models \alpha$.

LLE. Since $\alpha \leftrightarrow \beta$, $\mathtt{mc}(\mathtt{K},\alpha) = \mathtt{mc}(\mathtt{K},\beta)$. Furthermore, $\mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha) = \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\beta)$. We conclude that $\alpha \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$ implies $\beta \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$.

RW. If γ is inconsistent, then the proof is immediate. Else, let $\mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\gamma) = \{\mathtt{K}_1,\ldots,\mathtt{K}_n\}$. From what we supposed, for every $\mathtt{K}_i$ we have $\mathtt{K}_i \cup \{\gamma\} \models \alpha$. Since $\models \alpha \rightarrow \beta$ we have $\mathtt{K}_i \cup \{\gamma\} \models \beta$.

Cut. The case when α is inconsistent is trivial. In the rest of the proof we suppose that α is consistent. Since $\alpha \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \beta$, β is consistent as well. Suppose $\alpha \wedge \beta \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$ and $\alpha \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \beta$. By means of contradiction, suppose $\alpha \not\mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$. This means that there exists $\mathtt{K}_i \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$ s.t. $\mathtt{K}_i \cup \{\alpha\} \not\models \gamma$. Observe that $\mathtt{K}_i \cup \{\alpha\} \models \beta$. Note that $\mathtt{K}_i \cup \{\alpha \wedge \beta\}$ is consistent and that $\mathtt{K}_i \cup \{\alpha \wedge \beta\} \not\models \gamma$. Hence, $\mathtt{K}_i \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$. Let us prove that there exists $\mathtt{K}_m \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$ s.t. $\mathtt{K}_m \succ_{\mathtt{K}} \mathtt{K}'_i$.
Case 1: $\mathtt{K}_i \in \mathtt{mc}(\mathtt{K},\alpha \wedge \beta)$. Since $\mathtt{K}_i \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$, there exists $\mathtt{K}_m \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$ s.t. $\mathtt{K}_m \succ_{\mathtt{K}} \mathtt{K}_i$.
Case 2: $\mathtt{K}_i \notin \mathtt{mc}(\mathtt{K},\alpha \wedge \beta)$. Let $\mathtt{K}_l \in \mathtt{mc}(\mathtt{K},\alpha \wedge \beta)$ be s.t. $\mathtt{K}_i \subseteq \mathtt{K}_l$. Since $\succeq_{\mathtt{K}}$ is a monotonic selection relation, $\mathtt{K}_l \succ_{\mathtt{K}} \mathtt{K}_i$. Let $\mathtt{K}_m \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$. Then, $\mathtt{K}_m \succeq_{\mathtt{K}} \mathtt{K}_l$. From transitivity of $\succeq_{\mathtt{K}}$, $\mathtt{K}_m \succ_{\mathtt{K}} \mathtt{K}_i$.
In both cases, there exists $\mathtt{K}_m \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$ s.t. $\mathtt{K}_m \succ_{\mathtt{K}} \mathtt{K}_i$. Observe that $\mathtt{K}_m \cup \{\alpha\}$ is consistent. Let us show that $\mathtt{K}_i \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$.
Case 1: $\mathtt{K}_m \in \mathtt{mc}(\mathtt{K},\alpha)$. Then $\mathtt{K}_i \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$.
Case 2: $\mathtt{K}_m \notin \mathtt{mc}(\mathtt{K},\alpha)$. Let $\mathtt{K}_n \in \mathtt{mc}(\mathtt{K},\alpha)$ be s.t. $\mathtt{K}_m \subsetneq \mathtt{K}_n$. Since $\succeq_{\mathtt{K}}$ is a monotonic selection relation, $\mathtt{K}_n \succ_{\mathtt{K}} \mathtt{K}_m \succ_{\mathtt{K}} \mathtt{K}_i$.
In both cases $\mathtt{K}_i \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$. Contradiction. So it must be that $\mathtt{K}_i \cup \{\alpha\} \models \gamma$.

Or. The case when $\alpha \vee \beta$ is inconsistent is trivial; in the rest of the proof we suppose that $\alpha \vee \beta$ is consistent. Towards a contradiction, let $\alpha \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$ and $\beta \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$ and suppose that $\alpha \vee \beta \not\mid\!\sim^{\mathtt{mc}}_{\mathtt{K}} \gamma$. Let $\mathtt{K}_i \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \vee \beta)$ be such that $\mathtt{K}_i \cup \{\alpha \vee \beta\} \not\models \gamma$. Let m be a model of $\mathtt{K}_i$, i.e. $m \models \mathtt{K}_i$, such that $m \not\models \gamma$. It must be $m \models \alpha$ or $m \models \beta$. Without loss of generality, suppose $m \models \alpha$. Observe that $\mathtt{K}_i \cup \{\alpha\}$ is consistent.
Let us prove that $\mathtt{K}_i \in \mathtt{mc}(K,\alpha)$ by using reduction ad absurdum. Thus, let us start by supposing that there exists $\mathtt{K}'_i \subseteq \mathtt{K}$ such that $\mathtt{K}'_i \cup \{\alpha\}$ is consistent and $\mathtt{K}_i \subsetneq \mathtt{K}'_i$. Observe that $\mathtt{K}'_i \cup \{\alpha \vee \beta\}$ is consistent. Contradiction with $\mathtt{K}_i \in \mathtt{mc}(\mathtt{K},\alpha \vee \beta)$. Thus it must be that $\mathtt{K}_i \in \mathtt{mc}(\mathtt{K},\alpha)$.

Let us now show that $\mathtt{K}_i \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$. Again, towards a contradiction, let us suppose that $\mathtt{K}_i \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$. Hence, there exists $\mathtt{K}'_i \in \mathtt{mc}(\mathtt{K},\alpha)$ such that $\mathtt{K}'_i \succ_{\mathtt{K}} \mathtt{K}'_i$. Observe that $\mathtt{K}'_i \cup \{\alpha \vee \beta\}$ is consistent (since $\mathtt{K}'_i \cup \{\alpha\}$ is consistent and $\alpha \models \alpha \vee \beta$). Let us show that there exists $\mathtt{K}_m \in \mathtt{mc}(\mathtt{K},\alpha \vee \beta)$ s.t. $\mathtt{K}_m \succ \mathtt{K}_i$.
Case 1: $\mathtt{K}'_i \in \mathtt{mc}(\mathtt{K},\alpha \vee \beta)$ is obvious.
Case 2: $\mathtt{K}'_i \notin \mathtt{mc}(\mathtt{K},\alpha \vee \beta)$. Then, there exists $\mathtt{K}_n \in \mathtt{mc}(\mathtt{K},\alpha \vee \beta)$ s.t. $\mathtt{K}'_i \subsetneq \mathtt{K}_n$. Since $\succeq_{\mathtt{K}}$ is a monotonic selection relation, $\mathtt{K}_n \succ_{\mathtt{K}} \mathtt{K}'_i \succ_{\mathtt{K}} \mathtt{K}_i$.
In both cases, $\mathtt{K}_i \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \vee \beta)$. Contradiction; thus it must be $\mathtt{K}_i \in \mathtt{mc}(\mathtt{K},\alpha)$. Observe that $m \models \mathtt{K}_i$ and $m \not\models \gamma$. Thus, $\mathtt{K}_i \not\models \gamma$. Consequently, $\alpha \not\mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$. Contradiction. So it must be that $\alpha \vee \beta \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$.

CM. We skip the trivial case when $\alpha \wedge \beta$ is inconsistent. Suppose $\alpha \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \beta$ and $\alpha \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$. Towards a contradiction, suppose there exists $\mathtt{K}_j \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$ such that $\mathtt{K}_j \cup \{\alpha \wedge \beta\} \not\models \gamma$. Observe that $\mathtt{K}_j \cup \{\alpha\}$ is consistent and that $\mathtt{K}_j \cup \{\alpha\} \not\models \gamma$. Thus, $\mathtt{K}_j \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$.

Let us prove that there exists $\mathtt{K}_l \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$ such that $\mathtt{K}_l \succ_{\mathtt{K}} \mathtt{K}_j$.
Case 1: $\mathtt{K}_j \in \mathtt{mc}(\mathtt{K},\alpha)$. Since $\mathtt{K}_j \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$, there exists $\mathtt{K}_l \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$ such that $\mathtt{K}_l \succ_{\mathtt{K}} \mathtt{K}_j$.
Case 2: $\mathtt{K}_j \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$. Like in the previous parts of the proof, we show that there exists $\mathtt{K}_n \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$ such that $\mathtt{K}_n \succ_{\mathtt{K}} \mathtt{K}_j$. So we proved that there exists $\mathtt{K}_l \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha)$ such that $\mathtt{K}_l \succ_{\mathtt{K}} \mathtt{K}_j$.
Observe that $\mathtt{K}_l \cup \{\alpha\} \models \beta$, thus $\mathtt{K}_l \cup \{\alpha \wedge \beta\}$ is consistent. Let us show that $\mathtt{K}_l \in \mathtt{mc}(\mathtt{K},\alpha \wedge \beta)$. Towards a contradiction, suppose that there exists $\mathtt{K}_p \in \mathtt{mc}(\mathtt{K},\alpha \wedge \beta)$ s.t. $\mathtt{K}_l \subsetneq \mathtt{K}_p$. There must exist $\delta \in \mathtt{K}_p \setminus \mathtt{K}_l$. Since $\mathtt{K}_p \cup \{\alpha\}$ is consistent and $\mathtt{K}_l \subsetneq \mathtt{K}_p$, $\mathtt{K}_l \notin \mathtt{mc}(\mathtt{K},\alpha)$, contradiction. So it must be that $\mathtt{K}_l \in \mathtt{mc}(\mathtt{K},\alpha \wedge \beta)$. This means that $\mathtt{K}_j \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$, contradiction. Thus, for every $\mathtt{K}_s \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K},\alpha \wedge \beta)$, $\mathtt{K}_s \cup \{\alpha \wedge \beta\} \models \gamma$. Consequently, it holds that $\alpha \wedge \beta \mid\!\sim^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}_{\mathtt{K}} \gamma$.

RM. We skip the trivial case when $\alpha \wedge \beta$ is inconsistent. Suppose $\alpha \mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} \gamma$ and $\alpha \not\mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} \neg\beta$ and let us prove $\alpha \wedge \beta \mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} \gamma$. Observe that for every $\mathtt{K}_l \in \mathtt{mc}(\mathtt{K}, \alpha)$ we have $\mathtt{K}_l \in \mathtt{mc}(\mathtt{K}, \alpha \wedge \beta)$ if and only if $\mathtt{K}_l \cup \{\alpha\} \not\models \neg\beta$. Since $\alpha \not\mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} \neg\beta$, there exists $\mathtt{K}_s \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha)$ such that $\mathtt{K}_s \cup \{\alpha\} \not\models \neg\beta$. Thus, $\mathtt{K}_s \in \mathtt{mc}(\mathtt{K}, \alpha \wedge \beta)$.

Let us show that $\mathtt{K}_s \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha \wedge \beta)$. Towards a contradiction, let $\mathtt{K}_j \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha \wedge \beta)$ be s.t. $\mathtt{K}_j \succ_{\mathtt{K}} \mathtt{K}_s$. Observe that $\mathtt{K}_j \cup \{\alpha\}$ is consistent. Let us prove that there exists $K_l \in \mathtt{mc}(\mathtt{K}, \alpha)$ such that $\mathtt{K}_l \succ_{\mathtt{K}} \mathtt{K}_s$. If $K_j \in \mathtt{mc}(\mathtt{K}, \alpha)$, the proof is over. Else, let $\mathtt{K}_n \in \mathtt{mc}(\mathtt{K}, \alpha)$ be such that $\mathtt{K}_j \subsetneq \mathtt{K}_n$. Since $\succeq_{\mathtt{K}}$ is a monotonic selection relation, $\mathtt{K}_n \succ_{\mathtt{K}} \mathtt{K}_j$. Thus $\mathtt{K}_n \succ_{\mathtt{K}} \mathtt{K}_s$. We conclude that $\mathtt{K}_s \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha)$. Contradiction. So it must be $\mathtt{K}_s \in \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha \wedge \beta)$.
Let us prove that for every $\mathtt{K}_j \in \mathtt{mc}(\mathtt{K}, \alpha \wedge \beta)$, if $\mathtt{K}_j \not\models \gamma$ then $\mathtt{K}_j \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha \wedge \beta)$. Since $\alpha \mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} \gamma$, $\mathtt{K}_j \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha)$. Since $\mathtt{K}_j \cup \{\alpha\}$ is consistent and $\succeq_{\mathtt{K}}$ is a monotonic selection relation, by using a similar reasoning as before, we obtain that $\mathtt{K}_s \succ_{\mathtt{K}} \mathtt{K}_j$. Hence, $\mathtt{K}_j \notin \mathtt{mc}_{\succeq_{\mathtt{K}}}(\mathtt{K}, \alpha \wedge \beta)$. This means that $\alpha \wedge \beta \mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} \gamma$. □

As a corollary, we get that:

Corollary 1. *Let* $\mathtt{K}$ *be a belief base,* $\oplus$ *an aggregation operator satisfying Composition, Decomposition, Symmetry and Monotonocity and let* s *be any scoring function. Then* $\mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq^s_{\mathtt{K},\oplus}}}$ *is rational.*

6 A Characterization Result

We have shown that for each belief base $\mathtt{K}$ and each monotonic selection relation $\succeq_{\mathtt{K}}$, $\mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}}$ is rational. We now show the converse implication, namely, that for each rational relation $\mathrel{|\!\sim}$, there exists a belief base $\mathtt{K}$ and a monotonic selection relation $\succeq_{\mathtt{K}}$ such that $\mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} = \mathrel{|\!\sim}$. More formally:

Proposition 3. *For every rational relation* $\mathrel{|\!\sim}$ *defined on a logical language* $\mathcal{L}$ *built over a finite set* $\mathcal{V}$ *of propositional variables, there exists a belief base* $\mathtt{K} \subseteq \mathcal{L}$ *and a monotonic selection relation* $\succeq_{\mathtt{K}}$ *such that* $\mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} = \mathrel{|\!\sim}$.

Proof. Let $\mathcal{V} = \{x_1, \ldots, x_n\}$. Since $\mathrel{|\!\sim}$ is a rational relation, we know from Theorem 5 by Lehmann and Magidor (1992) that there exists a ranked model $W = \langle V, l, \prec\rangle$ defining this relation. Let $\mathtt{K} = \{w_1, \ldots, w_{2^n}\}$ be the base composed of all corresponding complete formulae, i.e.,
$w_1 = \neg x_1 \wedge \neg x_2 \wedge \ldots \wedge \neg x_n$
$w_2 = \neg x_1 \wedge \neg x_2 \wedge \ldots \wedge \quad x_n$
$\ldots$
$w_{2^n} = x_1 \wedge \quad x_2 \wedge \ldots \wedge \quad x_n$.

Let f be a mapping $f : \mathtt{K} \to V$ such that for every $w \in \mathtt{K}$, $v = f(w)$ if and only if $v \in V$ is a minimal with respect to $\prec$ state such that $l(v) \models w$.

Let us now define the relation $\succeq_K \subseteq 2^{\mathtt{K}} \times 2^{\mathtt{K}}$. Let $\mathtt{K}_i, \mathtt{K}_j \subseteq \mathtt{K}$. If $|\mathtt{K}_i| > |\mathtt{K}_j|$, let $\mathtt{K}_i \succ_{\mathtt{K}} K_j$. If $|\mathtt{K}_i| = |\mathtt{K}_j| \neq 1$, let $\mathtt{K}_i \succeq_{\mathtt{K}} K_j$ and $\mathtt{K}_j \succeq_{\mathtt{K}} K_i$. If $|\mathtt{K}_i| = |\mathtt{K}_j| = 1$, we proceed as follows. Let $\mathtt{K}_i = \{w'\}$ and $\mathtt{K}_j = \{w''\}$. Let $v' = f(w')$ and $v'' = f(w'')$. If $v' \prec v''$, let $\mathtt{K}_i \succ_{\mathtt{K}} \mathtt{K}_j$. If $v'' \prec v'$, let $\mathtt{K}_j \succ_{\mathtt{K}} \mathtt{K}_i$. Else, it must be that the two states are equally preferred. Let $\mathtt{K}_i \succeq_{\mathtt{K}} \mathtt{K}_j$ and $\mathtt{K}_j \succeq_{\mathtt{K}} K_i$. It is obvious that $\succeq_{\mathtt{K}}$ is a monotonic selection relation. Also $\alpha \mathrel{|\!\sim} \beta$ if and only if $\alpha \mathrel{|\!\sim}_W \beta$, where $\mathrel{|\!\sim}_W$ is the relation defined by W [Lehmann and Magidor, 1992]. Also, it is immediate to see that $\alpha \mathrel{|\!\sim}_W \beta$ if and only if $\alpha \mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} \beta$, which concludes the proof. □

Putting the last two results together, we obtain a new representation theorem for rational inference relations:

Theorem 1. *A relation* $\mathrel{|\!\sim}$ *is rational if and only if there exists a belief base* $\mathtt{K}$ *and a monotonic selection relation* $\succeq_{\mathtt{K}}$ *such that* $\mathrel{|\!\sim}_{\mathtt{K}}^{\mathtt{mc}_{\succeq_{\mathtt{K}}}} = \mathrel{|\!\sim}$.

7 Conclusion

In this paper we have pointed out a new family of inference relations from inconsistent belief bases based on the selection of maximal consistent subsets. This selection leads to inference relations with a stronger inferential power than the basic relation based on all maximal consistent subsets. We have provided examples of inference relations from this family which preserve more consequences than the basic relation. The selection process must obey some principles to ensure that it gives rise to an inference relation which is preferential. We have defined a general class of monotonic selection relations for comparing maximal consistent sets that ensures this property. It turns out that this class corresponds precisely to the class of rational inference relations. Thus, we have obtained a new representation theorem for the class of rational inference relations in terms of deduction from (selections of) maximal consistent subsets.

The work presented in this paper is related to a number of approaches where inference is defined from maximal subsets of defaults (see e.g., [Reiter, 1980; Poole, 1988; Makinson, 2005]). Some propositions reported by Gärdenfors and Makinson (1994) are quite close to results presented in this paper. Indeed, one can find several characterization results for nonmonotonic inference operators based on selection functions over maximal consistent subsets in their paper (see also [Freund, 1998] for related results on the links between extensions and preferential inference). The work by Gärdenfors and Makinson (1994) nevertheless departs from our own one from several aspects. Thus, while we focus on finite classical propositional logic, Gärdenfors and Makinson considered more general logical systems. Their representation theorem for rational inference also requires the base to be consistent and deductively closed, while in our approach, the base is finite, so it is not deductively closed, and it can be inconsistent.

A perspective for further research consists in determining the extent to which our results can be lifted from the flat case to the prioritized case. Indeed, though only few inference principles have been defined so far for dealing with inconsistent, flat bases $\mathtt{K}$ (as advocated in the introduction), there has been an abundant literature about the design of inference relations from preferred subbases of $\mathtt{K}$, where the preference relation is induced from a plausibility ordering over the formulae in $\mathtt{K}$ (see e.g., [Brewka, 1989; Nebel, 1991; Benferhat *et al.*, 1998]). It would be interesting to study how to combine the selection principles at work in such approaches with monotonic selection relations to give rise to new rational inference relations in the prioritized case.

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