New Inference Relations from Maximal Consistent Subsets

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Abstract

Given an inconsistent, flat belief base, we show how to draw non-trivial conclusions from it by selecting some of its maximal consistent subsets. This selection leads to inference relations with a stronger inferential power than the one based on all maximal consistent subsets, without questioning the fact that they are preferential relations (in the sense of KLM).

Introduction

Consider a scenario where a propositional, finite belief base $\mathtt{K}$ is obtained by putting together in a common repository some pieces of information, issued possibly from several sources, and represented by propositional formulas. Suppose also that the origins of the pieces of information (e.g., the sources they come from) are unknown or have been lost (which is a common assumption underlying for instance the AGM setting for belief revision, where it is supposed as well that one cannot trace back the pieces of knowledge). For the sake of illustration, suppose that the (contradictory) pieces of information that were gathered concern an incoming model of car from our favorite brand (those pieces of information have been obtained through different sources, including car magazines, websites, friends, etc.), Let $\mathtt{K} = \{\varphi_1, \ldots, \varphi_6\}$, where the formulas in it state that the new car has a 6-cylinder engine: $\varphi_1 = e6c$, that it has a manual gearbox: $\varphi_2 = mg$, that it has a turbo (t) and that a car with a 6-cylinder engine is not a sport car: $\varphi_3 = t \wedge (e6c \rightarrow \neg sc)$, that the car has a low fuel consumption (lc) and does not have a manual gearbox: $\varphi_4 = lc \wedge \neg mg$, that it is a sport car (sc), that does not have a manual gearbox and that a sport car does not have a low fuel consumption, and that a car with a 6-cylinder engine does not have 4-wheel drive ($\mathsf{wd4}$): $\varphi_5 = sc \wedge \neg mg \wedge (sc \rightarrow \neg lc) \wedge (e6c \rightarrow \neg wd4)$, that the car has 4-wheel drive, does not have a manual gearbox, and that cars with 4-wheel drive do not have a low fuel consumption, and that cars with a 6-cylinder engine do not need a turbo: $\varphi_6 = wd4 \wedge \neg mg \wedge (wd4 \rightarrow \neg lc) \wedge (e6c \rightarrow \neg t)$.

It can be easily checked that $\mathtt{K}$ has 5 maximal consistent subsets, namely: $\mathtt{K}_1 = \{\varphi_1, \varphi_6\}$, $\mathtt{K}_2 = \{\varphi_1, \varphi_5\}$, $\mathtt{K}_3 = \{\varphi_1, \varphi_2, \varphi_3\}$, $\mathtt{K}_4 = \{\varphi_1, \varphi_3, \varphi_4\}$, $\mathtt{K}_5 = \{\varphi_3, \varphi_5, \varphi_6\}$.

Using skeptical inference from all these maximal consistent subsets, none of the six formulas in $\mathtt{K}$ can be derived as a conclusion. This means that for every formula φ_i (with $i \in \{1, \ldots, 6\}$), there exists a maximal consistent subset M_j ($j \in \{1, \ldots, 5\}$) of $\mathtt{K}$ such that $M_j \models \neg\varphi_i$. As a consequence, only very weak conclusions composed of disjunctions of those formulas can be obtained as consequences. However, all the incoming pieces of information φ_i do not play symmetric role with respect to the maximal consistent subsets M_j. Consider for instance φ_1 and φ_2: we have that φ_1 is a logical consequence of 4 (over 5) maximal consistent subsets of $\mathtt{K}$, while φ_2 is a logical consequence of only one of them. Since the global inconsistency of a belief base is often due to the presence in it of erroneous pieces of information, it makes sense to take advantage of this discrepancy to consider some pieces of information as more reliable than others because they are more consensual / less conflicting with the other pieces of information which have been reported.

Inference from Selected Maximal Consistent Subsets

Our main objective is to identify selection criteria on maximal consistent subsets, leading to preserve more information from the belief bases while guaranteeing that the induced inference relations are preferential ones (Kraus, Lehmann, and Magidor 1990). One already knows that such criteria exist, since the inference relation based on the selection of *cardinality*-maximal consistent subsets of $\mathtt{K}$ is preferential (Benferhat et al. 1993), but we would like to identify other inference schemas and more general conditions on the set of all maximal consistent subsets which are sufficient to ensure that the corresponding inference relations are preferential ones.

To this end, we introduce new inference relations based on a selection of the maximal consistent subsets of $\mathtt{K}$ maximizing a given scoring function. Let us note $\mathtt{mc}(\mathtt{K})$ the set of all maximal (w.r.t. set inclusion) consistent subsets of $\mathtt{K}$.

Definition 1 ($\mathtt{mc}_{\mathtt{score}}$)**.** *Given a belief base* $\mathtt{K}$*, a formula* α*, and a real-valued mapping* $\mathtt{score}$*, we define* $\mathtt{mc}_{\mathtt{score}}(K, \alpha) = \{\mathtt{K}_i \in \mathtt{mc}(\mathtt{K} \cup \{\alpha\}) \mid \alpha \in \mathtt{K}_i$ *and there exists no* $\mathtt{K}'_i \in \mathtt{mc}(\mathtt{K} \cup \{\alpha\})$ *such that* $\alpha \in \mathtt{K}'_i$ *and* $\mathtt{score}(\mathtt{K}'_i) > \mathtt{score}(\mathtt{K}_i)\}$.

Definition 2 (Inference from selected subsets)**.** *Let* $\mathtt{K}$ *be* *a belief base and* α *and* β *be two formulas. We say that* $\alpha \mid\!\sim_{\mathtt{K}}^{\mathtt{mc}_{\mathtt{score}}} \beta$ *if and only if either* α *is inconsistent, or for every* $S \in \mathtt{mc}_{\mathtt{score}}(K, \alpha)$ *we have* $S \models \beta$.

Here is a first inference relation of this family, based on the number of maximal consistent sets a formula belongs to. We say that a formula φ is trivial iff $\varphi \equiv \top$ or $\varphi \equiv \bot$.

Definition 3 ($\mathtt{score}_{\mathtt{K}}^{\#\mathtt{mc}}$)**.** *Let* K *be a belief base and* $\alpha \in \mathtt{K}$. *We define*

$$\mathtt{score}_{\mathtt{K}}^{\#\mathtt{mc}}(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is trivial} \\ |\{\mathtt{K}_i \in \mathtt{mc}(\mathtt{K}) \mid \alpha \in \mathtt{K}_i\}| & \text{otherwise} \end{cases}$$

We also introduce the following definition:

Definition 4 ($\mathtt{score}_{\mathtt{K},sum}^{\#mc}$)**.** *Let* K *be a belief base, and let* $\mathtt{K}_i \subseteq \mathtt{K}$. *We define*

$$\mathtt{score}_{\mathtt{K},sum}^{\#mc}(\mathtt{K}_i) = \sum_{\alpha \in \mathtt{K}_i} \mathtt{score}_{\mathtt{K}}^{\#\mathtt{mc}}(\alpha).$$

Applying this relation to the example reported in the introduction, we obtain that the best maximal consistent subsets of K are $\mathtt{K}_3$ and $\mathtt{K}_4$. Therefore, we get that $\top \mid\!\sim_{\mathtt{K},sum}^{\#\mathtt{mc}} e6c \wedge t \wedge \neg sc$. Hence, we conclude that the car has a 6-cylinder engine, a turbo, and is not a sport car.

Another valuable inference relation is obtained by considering the number of minimal inconsistent sets to define the score of each formula.

Definition 5 ($\mathtt{score}_{\mathtt{K}}^{\#\mathtt{mus}}$)**.** *Let* K *be a belief base and* $\alpha \in \mathtt{K}$. *Let us note* $\mathtt{mus}(\mathtt{K})$ *the set of all minimal (w.r.t. set inclusion) inconsistent subsets of* K. *We define*

$$\mathtt{score}_{\mathtt{K}}^{\#\mathtt{mus}}(\alpha) = \begin{cases} 0 \text{ if } \alpha \text{ is trivial, otherwise} \\ 1 + |\mathtt{mus}(\mathtt{K})| - \\ \quad |\{\mathtt{K}_i \subseteq \mathtt{K} \mid \mathtt{K}_i \in \mathtt{mus}(\mathtt{K}), \alpha \in \mathtt{K}_i\}| \end{cases}$$

Yet another inference relation takes advantage of the inconsistency value MIV (Hunter and Konieczny 2010):

Definition 6 (MIV)**.** *Let* K *be a belief base and* $\alpha \in \mathtt{K}$. *We define* $\mathtt{MIV}_{\mathtt{K}}(\alpha) = \sum_{M \in \mathtt{mus}(\mathtt{K}), \alpha \in M} \frac{1}{|M|}$.

Definition 7 ($\mathtt{score}_{\mathtt{K}}^{\#\mathtt{miv}}$)**.** *Let* K *be a belief base and* $\alpha \in \mathtt{K}$. *We define* $\mathtt{max}_{\mathtt{miv}}(\mathtt{K}) = max_{\alpha \in \mathtt{K}} \mathtt{MIV}_{\mathtt{K}}(\alpha)$, *and*

$$\mathtt{score}_{\mathtt{K}}^{\#\mathtt{miv}}(\alpha) = \begin{cases} 0 \text{ if } \alpha \text{ is trivial, otherwise} \\ 1 + \mathtt{max}_{\mathtt{miv}}(\mathtt{K}) - \mathtt{MIV}_{\mathtt{K}}(\alpha) \end{cases}$$

Note that though the three scoring functions above lead to select the same maximal consistent subsets for the running example, this is not the case in general.

Properties

Let us now show that the three inference relations defined above, and other ones based on the same construction, have desirable logical properties. In the following, the set of positive integers is denoted by $\mathbb{N}$ and the set of non-negative real numbers by $\mathbb{R}_{\geq 0}$.

Definition 8 ($\mathtt{score}_{\mathtt{K}}$)**.** *Given a belief base* K, $\mathtt{score}_{\mathtt{K}} : \mathtt{K} \to \mathbb{R}_{\geq 0}$ *is a* scoring function *if* $\mathtt{score}_{\mathtt{K}}(\alpha) = 0$ *if and only if* α *is a trivial formula.*

We now need to compare sets of formulas based on scores of individual formulas. This calls for aggregation functions.

Definition 9 (Aggregation function)**.** *We say that* $\oplus$ *is an* aggregation function *if for every* $n \in \mathbb{N}$, *for every* $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$ $\oplus(x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}$.

Definition 10 (Properties of aggregation functions)**.** *Let* $\vec{x}$ *be a shortcut for* $x_1, \ldots, x_n$. *An aggregation function* $\oplus$ *satisfies*

- *Composition if* $\oplus(\vec{x}) \leq (\vec{y})$ *implies* $\oplus(\vec{x}, z) \leq (\vec{y}, z)$
- *Decomposition if* $\oplus(\vec{x}, z) \leq (\vec{y}, z)$ *implies* $\oplus(\vec{x}) \leq (\vec{y})$
- *Symmetry if for every permutation* σ, $\oplus(\vec{x}) = \oplus(\sigma(\vec{x}))$
- *Monotonicity if for every* $z > 0$ *we have* $\oplus(\vec{x}, z) > \oplus(\vec{x})$

As an example of aggregation function satisfying all the above properties, consider sum, which returns the sum of all scores. Composition, Decomposition and Symmetry were introduced in (Konieczny, Lang, and Marquis 2004). We add here a new property, namely Monotonicity.

From a scoring function and an aggregation function, a relation for comparing the subsets of K can be easily defined:

Definition 11 ($\mathtt{score}_{\mathtt{K},\oplus}$)**.** *Let* K *be a belief base and* $\mathtt{K}_i = \{\alpha_1, \ldots, \alpha_n\}$ *a subset of* K. *Let* $\mathtt{score}_{\mathtt{K}}$ *be a scoring function and* $\oplus$ *an aggregation operator. We define*

$$\mathtt{score}_{\mathtt{K},\oplus}(\mathtt{K}_i) = \oplus(\mathtt{score}_{\mathtt{K}}(\alpha_1), \ldots, \mathtt{score}_{\mathtt{K}}(\alpha_n)).$$

Proposition 1. *Let* K *be a belief base,* $\oplus$ *an aggregation operator satisfying Composition, Decomposition, Symmetry and Monotonicity and let* $\mathtt{score}_{\mathtt{K}}$ *be any scoring function. Then* $\mid\!\sim_{\mathtt{K}}^{\mathtt{mc}_{\mathtt{score}_{K,\oplus}}}$ *is a preferential relation.*

Conclusion

In this paper we have pointed out new inference relations based on the selection of maximal consistent subsets of a belief base. By construction, those inference relations have a stronger inferential power than the one based on all maximal consistent subsets, but they are still preferential ones. We are currently looking for a more general characterization of the selection functions to be considered for ensuring that the induced inference relations are preferential ones.

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