

A Tableau Method for Generalized Qualitative Spatio-Temporal Reasoning

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Abstract

We study the spatiotemporal logic that results by combining the propositional temporal logic (PTL) with a qualitative spatial constraint language, namely, the \mathcal{L}_1 logic, and present a first semantic tableau method that given a \mathcal{L}_1 formula ϕ systematically searches for a model for ϕ . Our approach builds on Wolper’s tableau method for PTL, while the ideas provided can be carried to other tableau methods for PTL as well. Further, we prove the correctness of our tableau method for \mathcal{L}_1 using the constraint properties of *compactness* and *patchwork*, whose implication in qualitative spatiotemporal reasoning was recently introduced in literature.

1 Introduction

Time and *space* are fundamental cognitive concepts that have been the focus of study in many scientific disciplines, including Artificial Intelligence and, in particular, Knowledge Representation. Knowledge Representation has been quite successful in dealing with the concepts of time and space, and has developed formalisms that range from temporal and spatial databases [Story and Worboys, 1995], to quantitative models developed in computational geometry [Preparata and Shamos, 1985] and qualitative constraint languages and logical theories developed in qualitative reasoning [Wolper and Zakharyashev, 2003].

Towards constraint-based qualitative spatiotemporal reasoning, most of the work has relied on formalisms based on the propositional temporal logic (PTL), also known as linear temporal logic, and the qualitative spatial constraint language RCC-8 [Wolper and Zakharyashev, 2003; 2000]. PTL [Huth and Ryan, 2004] is the well known temporal logic comprising operators \mathcal{U} (until), \circ (next point in time), \square (always), and \diamond (eventually) over various flows in time, such as $\langle \mathbb{N}, < \rangle$. RCC-8 is a fragment of the Region Connection Calculus (RCC) [Randell *et al.*, 1992] and is used to describe regions that are non-empty regular subsets of some topological space by stating their topological relations to each other. The topological relations comprise relations *DC* (disconnected), *EC* (externally connected), *EQ* (equal), *PO* (partially overlapping), *TPP* (tangential proper part), *TPPi* (tangential

proper part inverse), *NTPP* (non-tangential proper part), *NTPPi* (non-tangential proper part inverse). These 8 relations are depicted in [Randell *et al.*, 1992, Fig. 4]. One of the most important of such formalisms is the \mathcal{ST}_1^- logic [Gabelaia *et al.*, 2003]. For example, one can have the following statement using that formalism: $\diamond TPP(X, Y)$, which translates to “eventually region X will be a tangential proper part of region Y ”.

In this paper, we consider a generalization of the \mathcal{ST}_1^- logic, denoted by \mathcal{L}_1 , which results by combining PTL [Huth and Ryan, 2004] with any qualitative spatial constraint language, such as RCC-8 [Randell *et al.*, 1992], Cardinal Direction Algebra (CDA) [Frank, 1991; Ligozat, 1998], and Block Algebra (BA) [Balbiani *et al.*, 2002], and present a first semantic tableau method that given a \mathcal{L}_1 formula ϕ systematically searches for a model for ϕ . This method builds on the tableau method for PTL of Wolper [Wolper, 1985], and makes use of the constraint properties of *compactness* and *patchwork* to ensure soundness and completeness. The implication of these properties in qualitative spatiotemporal reasoning was recently introduced in [Sioutis *et al.*, 2015]. It is important to note, that Wolper’s method serves as the basis to illustrate our line of reasoning, and that the techniques presented here can be carried to other more efficient tableau methods for PTL as well, such as the systematic semantic tableaux for PTL presented in [Gaintzarain *et al.*, 2008].

The organization of the paper is as follows. In Section 2 we recall the definition of a qualitative spatial constraint language, along with the properties of compactness and patchwork. Section 3 introduces the \mathcal{L}_1 logic, and explains its implication with compactness and patchwork. In Section 4 we present our tableau method for checking the satisfiability of a \mathcal{L}_1 formula. In Section 5 we conclude and give directions for future work.

2 Preliminaries

A (binary) qualitative temporal or spatial constraint language [Renz and Ligozat, 2005] is based on a finite set B of *jointly exhaustive and pairwise disjoint* (JEPD) relations defined on a domain D , called the set of base relations. The base relations of set B of a particular qualitative constraint language can be used to represent the definite knowledge between any two entities with respect to the given level of granularity. B contains the identity relation Id , and is closed un-

der the inverse operation ($^{-1}$). Indefinite knowledge can be specified by disjunctions of possible base relations, and is represented by the set containing them. Hence, 2^B represents the total set of relations. 2^B is equipped with the usual set-theoretic operations (union and intersection), the inverse operation, and the weak composition operation denoted by \diamond [Renz and Ligozat, 2005]. A network from any qualitative spatial constraint language, such as RCC-8 [Randell *et al.*, 1992], Cardinal Direction Algebra (CDA) [Frank, 1991; Ligozat, 1998], or Block Algebra (BA) [Balbiani *et al.*, 2002], can be formulated as a qualitative constraint network (QCN) as follows.

Definition 1 A QCN is a tuple (V, C) where V is a non-empty finite set of variables and C is a mapping that associates a relation $C(v, v') \in 2^B$ to each pair (v, v') of $V \times V$. Mapping C is such that $C(v, v) = \{\text{Id}\}$ and $C(v, v') = (C(v', v))^{-1}$ for every $v, v' \in V$.

If b is a base relation, $\{b\}$ is a singleton relation. An *atomic* QCN is a QCN where each constraint is a singleton relation. Note that we always regard a QCN as a complete network. Given two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$, $\mathcal{N} \cup \mathcal{N}'$ denotes the QCN $\mathcal{N}'' = (V'', C'')$, where $V'' = V \cup V'$, $C''(u, v) = C''(v, u) = B$ for all $(u, v) \in (V \setminus V') \times (V' \setminus V)$, $C''(u, v) = C(u, v) \cap C'(u, v)$ for every $u, v \in V \cap V'$, $C''(u, v) = C(u, v)$ for every $(u, v) \in (V \times V) \setminus (V' \times V')$, and $C''(u, v) = C'(u, v)$ for every $(u, v) \in (V' \times V') \setminus (V \times V)$. Given a QCN $\mathcal{N} = (V, C)$ and $u, v \in V$, $C(u, v)$ will be also denoted by $\mathcal{N}[u, v]$.

We can interpret any QCN $\mathcal{N} = (V, C)$ using a structure of the form $\mathcal{M}_S = (D, \alpha)$, where α is a mapping that associates elements of D to elements of V . For the case of RCC-8 for example, if \mathcal{T} is some topological space [Munkres, 2000], let $\mathcal{R}(\mathcal{T})$ denote the set of all non-empty regular closed subsets in \mathcal{T} . Then, the domain D of RCC-8 is the set $\mathcal{R}(\mathcal{T})$, which can be infinite. A structure $\mathcal{M}_S = (D, \alpha)$ is a model for a QCN $\mathcal{N} = (V, C)$, also called a *solution*, if mapping α can yield a spatial configuration where the relations between the spatial variables can be described by C . It follows that a QCN is satisfiable if there exists a model for it. A *partial solution* for \mathcal{N} on $V' \subseteq V$ is the mapping α restricted to V' .

Checking the satisfiability of a RCC-8, CDA, or BA network is NP-complete in the general case [Renz, 1999; Ligozat, 1998; Balbiani *et al.*, 2002]. However, there exist large maximal tractable subclasses of RCC-8, CDA, and BA, which allow for practical and efficient reasoning. In particular, checking the satisfiability of a QCN (V, C) of RCC-8, CDA, or BA comprising only relations from one of its maximal tractable subclasses containing all singleton relations and the universal relation B , can be done in $O(|V|^3)$ time using the \diamond -consistency algorithm (also called *algebraic closure*), that iteratively performs the following operation until a fixed point \bar{C} is reached: $\forall v, v', v'' \in V, C(v, v') \leftarrow C(v, v') \cap (C(v, v'') \diamond C(v'', v'))$ [Renz and Ligozat, 2005].

Let us now recall the definitions of the constraint properties of *patchwork* and *compactness* in the context of qualitative reasoning.

Definition 2 ([Lutz and Milicic, 2007]) A qualitative temporal or spatial constraint language has patchwork, if for

any finite satisfiable constraint networks $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$ defined in this language where for any $u, v \in V \cap V'$ we have that $C(u, v) = C'(u, v)$, the constraint network $\mathcal{N} \cup \mathcal{N}'$ is satisfiable.

In light of patchwork, which concerns finite networks, compactness ensures satisfiability of an infinite sequence of finite satisfiable extensions of a network.

Definition 3 ([Huang, 2012]) A qualitative temporal or spatial constraint language has compactness, if any infinite set of constraints defined in this language is satisfiable whenever all its finite subsets are satisfiable.

Intuitively, patchwork ensures that the combination of two satisfiable constraint networks that agree on their common part, i.e., on the constraints between their common variables, continues to be satisfiable, while compactness allows for defining satisfiable networks of infinite size.

3 The \mathcal{L}_1 spatiotemporal logic

In general, a spatial QCN, as described in Section 2, constitutes a static spatial configuration in some domain, over a set of spatial variables V . To be able to describe a spatial configuration that changes over time, we can combine PTL [Huth and Ryan, 2004] with a qualitative spatial constraint language in a unique formalism. The domain D of a QCN will always remain the same, but the spatial variables in it may spatially change with the passing time (e.g., in shape, size, or orientation). We can interpret formulas of such a spatiotemporal formalism using a spatiotemporal structure defined as follows.

Definition 4 A ST-structure is a tuple $\mathcal{M}_{ST} = (D, \mathbb{N}, \alpha)$, where α is a mapping that associates elements of D to the spatial variables of a set V at a point of time $i \in \mathbb{N}$. Thus, $\alpha(i)$ denotes the set of elements of D that are associated with the spatial variables of V at point of time i . By extending notation, $\alpha(v, i)$, where $v \in V$, denotes the element of D that is associated with spatial variable v at point of time i .

For example, in the case of RCC-8, α would be a mapping associating elements of $\mathcal{R}(\mathcal{T})$ to spatial region variables at a point of time $i \in \mathbb{N}$. The set of atomic propositions AP in the case of standalone PTL [Huth and Ryan, 2004] is replaced by the set of base relations B of the qualitative spatial constraint language considered. We will call such a spatiotemporal formula over B a \mathcal{L}_0 formula. Thus, the set of \mathcal{L}_0 formulas over B is inductively defined as follows: if $P \in B$ then P is a \mathcal{L}_0 formula, and if ψ and ϕ are \mathcal{L}_0 formulas then $\neg\phi$, $\phi \vee \psi$, $\circ\phi$, $\square\phi$, $\diamond\phi$, and $\phi \mathcal{U} \psi$ are \mathcal{L}_0 formulas.

A simple example of a \mathcal{L}_0 formula is $\square NTPP(\text{Athens}, \text{Greece})$, stating that Athens will always be located in Greece. To increase the expressiveness of the \mathcal{L}_0 logic we can allow the application of operator \circ to spatial variables, i.e., we can have the following statement in RCC-8: $\square EQ(\text{Greece}, \circ \text{Greece})$, which translates to ‘‘Greece will never change its borders’’. We call the enriched logic the \mathcal{L}_1 logic.

Definition 5 Given a \mathcal{L}_1 formula ϕ over B , we write $\langle \mathcal{M}_{ST}, i \rangle \models \phi$ for the fact that \mathcal{M}_{ST} satisfies ϕ at point of time i , with $i \in \mathbb{N}$ (or formula ϕ is true in \mathcal{M}_{ST} at point of time i). The semantics is then defined as follows:

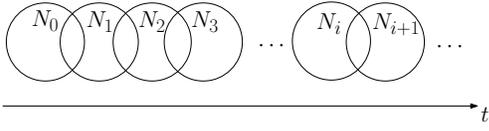


Figure 1: A countably infinite sequence of satisfiable atomic QCNs that agree on their common part

- $\langle \mathcal{M}_{\text{ST}}, i \rangle \models P(\circ^n v, \circ^m v')$ iff the relation that holds between $\alpha(v, i+n)$ and $\alpha(v', i+m)$ is the relation P , with $P \in \mathcal{B}$
- $\langle \mathcal{M}_{\text{ST}}, i \rangle \models \neg\phi$ iff $\langle \mathcal{M}_{\text{ST}}, i \rangle \not\models \phi$
- $\langle \mathcal{M}_{\text{ST}}, i \rangle \models \phi \vee \psi$ iff $\langle \mathcal{M}_{\text{ST}}, i \rangle \models \phi$ or $\langle \mathcal{M}_{\text{ST}}, i \rangle \models \psi$
- $\langle \mathcal{M}_{\text{ST}}, i \rangle \models \phi \mathcal{U} \psi$ if there exists a $k \in \mathbb{N}$ such that $i \leq k$, $\langle \mathcal{M}_{\text{ST}}, k \rangle \models \psi$, and for all $j \in \mathbb{N}$, if $i \leq j$ and $j < k$ then $\langle \mathcal{M}_{\text{ST}}, j \rangle \models \phi$

Formulas of the form $\diamond\phi$ and $\square\phi$ are abbreviations for $\top \mathcal{U} \phi$ and $\neg(\top \mathcal{U} \neg\phi)$ respectively. A structure $\mathcal{M}_{\text{ST}} = (D, \mathbb{N}, \alpha)$, for which $\langle \mathcal{M}_{\text{ST}}, 0 \rangle \models \phi$, is a model for ϕ . It follows that a \mathcal{L}_1 formula ϕ is satisfiable if there exists a model for it. Note that a formula of the form $\circ^k P(\circ^l v, \circ^m v')$ is equivalent to formula $P(\circ^{l+k} v, \circ^{m+k} v')$. The size of $P(\circ^{l+k} v, \circ^{m+k} v')$ is then defined to be equal to $\max\{l+k, m+k\}$. Like in [Balbiani and Condotta, 2002], we define the size of any \mathcal{L}_1 formula ϕ , denoted by $|\phi|$, inductively as follows: $P(\circ^l v, \circ^m v') = \max\{l, m\}$; $|\neg\phi| = |\phi|$; $|\phi \vee \psi| = |\phi \mathcal{U} \psi| = \max\{|\phi|, |\psi|\}$. The size of a set of \mathcal{L}_1 formulas $\chi = \{\phi, \psi, \dots\}$, will be the maximum size among its formulas, i.e., $|\chi| = \max\{|\phi|, |\psi|, \dots\}$. The number of occurrences of symbols in a \mathcal{L}_1 formula ϕ will be denoted by $\text{length}(\phi)$.

We then have the following results by [Sioutis *et al.*, 2015] with respect to the implication of patchwork and compactness in qualitative spatiotemporal reasoning:

Proposition 1 ([Sioutis *et al.*, 2015]) Let $V = \{v_0, \dots, v_n\}$ be a set of variables, $w \geq 0$ an integer, and $\mathcal{S} = (\mathcal{N}_0 = (V_0, C_0), \mathcal{N}_1 = (V_1, C_1), \dots)$ a countably infinite sequence of satisfiable atomic QCNs, as shown in Figure 1, such that:

- for each $i \geq 0$, V_i is defined by the set of variables $\{v_0^0, \dots, v_n^0, \dots, v_0^w, \dots, v_n^w\}$,
- for each $i \geq 0$, for all $m, m' \in \{0, \dots, n\}$, and for all $k, k' \in \{1, \dots, w\}$, $C_i(v_m^k, v_{m'}^{k'}) = C_{i+1}(v_m^{k-1}, v_{m'}^{k'-1})$.

We have that if the constraint language considered has compactness and patchwork for atomic QCNs, then \mathcal{S} defines a consistent set of qualitative constraints.

The second proposition follows.

Proposition 2 ([Sioutis *et al.*, 2015]) Let $V = \{v_0, \dots, v_n\}$ be a set of variables, $w \geq 0, t > t' \geq 0$ three integers, and $\mathcal{S} = (\mathcal{N}_0 = (V_0, C_0), \mathcal{N}_1 = (V_1, C_1), \dots)$ a countably infinite sequence of satisfiable atomic QCNs, as shown in Figure 2, such that:

- for each $i \geq 0$, V_i is defined by the set of variables $\{v_0^0, \dots, v_n^0, \dots, v_0^w, \dots, v_n^w\}$,
- for each $i \geq 0$, for all $m, m' \in \{0, \dots, n\}$, and for all $k, k' \in \{1, \dots, w\}$, $C_i(v_m^k, v_{m'}^{k'}) = C_{i+1}(v_m^{k-1}, v_{m'}^{k'-1})$,
- for all $m, m' \in \{0, \dots, n\}$ and all $k, k' \in \{0, \dots, w\}$, $C_{t'}(v_m^k, v_{m'}^{k'}) = C_t(v_m^k, v_{m'}^{k'})$.

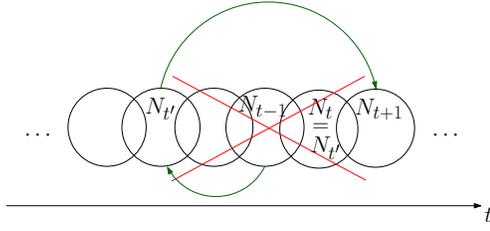


Figure 2: A countably infinite sequence of satisfiable atomic QCNs that contains a sub-sequence which begins and ends with two QCNs representing the same set of spatial constraints; we can reduce the sub-sequence to just considering the first QCN and patch it with the QCN following the sub-sequence

Let $\mathcal{S}' = (\mathcal{N}'_0 = (V'_0, C'_0), \mathcal{N}'_1 = (V'_1, C'_1), \dots)$ be the infinite sequence defined by:

- for all $i \in \{0, \dots, t'\}$, $\mathcal{N}'_i = \mathcal{N}_i$,
- for all $i > t'$, $V'_i = V_i$, and for all $m, m' \in \{0, \dots, n\}$ and all $k, k' \in \{0, \dots, w\}$, $C'_i(v_m^k, v_{m'}^{k'}) = C_{i+(t-t')}(v_m^k, v_{m'}^{k'})$.

We have that if the constraint language considered has compactness and patchwork for atomic QCNs, then \mathcal{S}' defines a consistent set of qualitative constraints.

The aforementioned propositions will be used to prove the correctness of our tableau method to be presented in the following section.

4 Semantic tableau for \mathcal{L}_1

In this section, we present a semantic tableau method that given a \mathcal{L}_1 formula ϕ systematically searches for a model for ϕ . The method builds on the tableau method for PTL of Wolper [Wolper, 1985], and makes use of the results presented in Section 3 to ensure soundness and completeness.

4.1 Rules for constructing a semantic tableau

The decomposition rules of the temporal operators are based on the following identities, which are called *eventualities* (where \square abbreviates $\neg\diamond\neg$):

- $\diamond\phi \equiv \phi \vee \circ\diamond\phi$
- $\phi \mathcal{U} \psi \equiv \psi \vee (\phi \wedge \circ(\phi \mathcal{U} \psi))$

Note that decomposing eventualities can lead to an infinite tableau. However, we will construct a finite tableau by identifying nodes that are labeled by the same set of formulas, thus, ensuring that infinite periodicity will not exist. To test a \mathcal{L}_1 formula ϕ for satisfiability, we will construct a directed graph. Each node n of the graph will be labeled by a set of formulas, and initially the graph will contain a single node, labeled by $\{\phi\}$. Similarly to Wolper [Wolper, 1985], we distinguish between *elementary* and *non-elementary* formulas:

Definition 6 A \mathcal{L}_1 formula is *elementary* if its main connective is \circ (viz., \circ -formula), or if it corresponds to a base relation $P \in \mathcal{B}$.

Then, the construction of the graph proceeds by using the following decomposition rules which map each non-elementary formula ϕ into a set of sets of formulas:

- $\neg P(\bigcirc^n v, \bigcirc^m v') \rightarrow \{\{P'(\bigcirc^n v, \bigcirc^m v')\} \mid P' \in B \setminus \{P\}\}$
- $\neg\neg\phi \rightarrow \{\{\phi\}\}$
- $\neg\bigcirc\phi \rightarrow \{\{\bigcirc\neg\phi\}\}$
- $\phi \wedge \psi \rightarrow \{\{\phi, \psi\}\}$
- $\neg(\phi \wedge \psi) \rightarrow \{\{\neg\phi\}, \{\neg\psi\}\}$
- $\diamond\phi \rightarrow \{\{\phi\}, \{\bigcirc\diamond\phi\}\}$
- $\neg\diamond\phi \rightarrow \{\{\neg\phi, \neg\bigcirc\diamond\phi\}\}$
- $\phi \mathcal{U} \psi \rightarrow \{\{\psi\}, \{\phi, \bigcirc(\phi \mathcal{U} \psi)\}\}$
- $\neg(\phi \mathcal{U} \psi) \rightarrow \{\neg\psi, \neg\phi \vee \neg\bigcirc(\phi \mathcal{U} \psi)\}$

During the construction, we *mark* formulas to which a decomposition rule has been applied to avoid decomposing the same formula twice. If ψ is a formula, ψ^* denotes ψ marked.

4.2 Systematic construction of a semantic tableau

A tableau \mathcal{T} can be seen as a directed graph where each of its nodes n is labeled with a set of formulas $\mathcal{T}(n)$. The root node is labeled with the singleton set $\{\phi\}$ for the \mathcal{L}_1 formula ϕ whose satisfiability we wish to check. The children of the nodes are obtained by applying the rules presented in Section 4.1.

Given a set of \mathcal{L}_1 formulas χ over the set of variables $\{x_0, \dots, x_l\}$, we denote by $expandVars(\chi)$ the set $\{\bigcirc^0 x_0, \dots, \bigcirc^0 x_l, \dots, \bigcirc^{|\chi|} x_0, \dots, \bigcirc^{|\chi|} x_l\}$. We first define a translation of a node of a tableau to a QCN.

Definition 7 Let n be a node of a tableau \mathcal{T} for a \mathcal{L}_1 formula ϕ , and $\{x_0, \dots, x_l\}$ the set of variables in ϕ . Then, $\mathcal{N}(n)$ will denote the QCN $= (V, C)$, where $V = \{v_0^0, \dots, v_l^0, \dots, v_0^{|\phi|}, \dots, v_l^{|\phi|}\}$, and $C(v_m^k, v_{m'}^{k'}) = \{P(\bigcirc^k x_m, \bigcirc^{k'} x_{m'})\}$ if $P(\bigcirc^k x_m, \bigcirc^{k'} x_{m'}) \in \mathcal{T}(n)$, and $C(v_m^k, v_{m'}^{k'}) = (B$ if $v_m^k \neq v_{m'}^{k'}$, else $\{Id\})$ otherwise, $\forall m, m' \in \{0, \dots, l\}$ and $\forall k, k' \in \{0, \dots, |\phi|\}$.

Let us also define the notions of a *state* and a *pre-state*, which we will be referring to a lot in what follows.

Definition 8 A node n that contains only elementary and marked formulas and for which we have that $\mathcal{N}(n)$ is atomic is called a *state*, and a node m that is either the root node or the direct child node of a state (which leaps to the next point of time) is called a *pre-state*.

We give a definition of eventuality fulfillment that will be of use later on.

Definition 9 Let \mathcal{T} be a tableau, and π a path in \mathcal{T} defined from nodes n_1, n_2, \dots, n_j . Any eventuality \diamond_{ϵ_2} or $\epsilon_1 \mathcal{U} \epsilon_2 \in \mathcal{T}(n_i)$, with $1 \leq i \leq j$, is fulfilled in π if there exists k , with $i \leq k \leq j$, such that $\epsilon_2 \in \mathcal{T}(n_k)$.

We now present Clotho, an algorithm that constructs a semantic tableau \mathcal{T} for a given formula ϕ , as shown in Algorithm 1. At any given point of time, we construct all the possible atomic QCNs comprising base relations that extend from the given point of time to a future point of time. This is achieved by repeatedly applying the decomposition rules to a node comprising unmarked non-elementary formulas (lines 4 to 9), and sequentially populating a node comprising only elementary and marked formulas with the universal relation B (lines 10 to 19) so that it may lead to a state. The universal relation B is only introduced on a pair of variables, if there does

Algorithm 1: Clotho(ϕ)

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in : A  $\mathcal{L}_1$  formula  $\phi$ .
output : A semantic tableau  $\mathcal{T}$  for  $\phi$ .
1 begin
2   create root node  $\{\phi\}$  and mark it unprocessed;
3   while  $\exists$  unprocessed node  $n$  do
4     if  $\mathcal{T}(n)$  contains an unmarked non-elementary
       formula  $\psi$  then
5       mark node  $n$  processed;
6       foreach  $\gamma \in \Gamma$ , where  $\Gamma$  is the result of applying a
           decomposition rule to  $\psi$  do
7         create a child node  $m$ ;
8          $\mathcal{T}(m) \leftarrow (\mathcal{T}(n) - \{\psi\}) \cup \gamma \cup \{\psi^*\}$ ;
9         mark node  $m$  unprocessed;
10      else if  $\mathcal{T}(n)$  contains only elementary and marked
          formulas then
11        mark node  $n$  processed;
12        filling  $\leftarrow \emptyset$ ;
13        foreach  $u, v \in expandVars(\phi)$  do
14          if  $\nexists P(u, v) \in \mathcal{T}(n)$  then
15            filling  $\leftarrow$  filling  $\cup \{B(u, v)\}$ ;
16          if filling  $\neq \emptyset$  then
17            create a child node  $m$ ;
18             $\mathcal{T}(m) \leftarrow \mathcal{T}(n) \cup$  filling;
19            mark node  $m$  unprocessed;
20          else if  $\mathcal{T}(n)$  contains  $\bigcirc$ -formulas then
21            create a child node  $m$ ;
22             $\mathcal{T}(m) \leftarrow \{\psi \mid \bigcirc\psi \in \mathcal{T}(n)\}$ ;
23             $\mathcal{T}(m) \leftarrow \mathcal{T}(m) \cup \{P(\bigcirc^{i-1}u, \bigcirc^{j-1}v) \mid$ 
24               $P(\bigcirc^i u, \bigcirc^j v) \in \mathcal{T}(n) \text{ if } i, j \geq 1\}$ ;
              mark node  $m$  unprocessed;

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not exist any base relation on that same pair. The universal relation B, as well as any other relation $r \in 2^B$, is essentially the disjunction of base relations, as noted in Section 2. In particular, B is the disjunction of all the base relations of a given qualitative constraint language. As such, by decomposing B into base relations using the disjunctive tableau rule, this approach allows us to obtain one or more nodes harboring atomic QCNs for a given point of time (viz., states), that represent a set of atomic spatial constraints in a fixed-width window of time. Once we have obtained our atomic QCNs for a given point of time, and assuming that the states that harbor them contain \bigcirc -formulas, we can leap to the next point of time and create pre-states, including all the atomic spatial constraints of the aforementioned QCNs that extend from the new point of time to a future point of time (lines 20 to 24). This can be seen as making a +1 time shift and maintaining all possible knowledge offered by previous states that extends from the new point of time to a future point of time. It is important to note that when we create a child node m of a node n (lines 7, 17, and 21), we only create a new node if there does not already exist a node in the graph labeled by $\mathcal{T}(m)$. Otherwise, we just create an arc from node n to the existing node.

Lemma 1 Let \mathcal{T} be a tableau for a \mathcal{L}_1 formula ϕ that has resulted after the application of algorithm Clotho. Then, \mathcal{T} is finite. Actually, if ϕ is over a set of l variables, then \mathcal{T} has at

Algorithm 2: Atropos(\mathcal{T})

in : A semantic tableau \mathcal{T} .
output : True or False.

```

1 begin
2   do
3     flag  $\leftarrow$  False;
4     if there is a node  $n$  such that  $\mathcal{N}(n)$  is an unsatisfiable
      QCN then
5       | eliminate node  $n$ ; flag  $\leftarrow$  True;
6     if all the children of a node  $n$  have been eliminated
      then
7       | eliminate node  $n$ ; flag  $\leftarrow$  True;
8     if a node  $n$  is a pre-state and not Lachesis( $\mathcal{T}, n$ ) then
9       | eliminate node  $n$ ; flag  $\leftarrow$  True;
10    while flag;
11    if  $\nexists$  node  $n \in \mathcal{T}$  then return False else return True;

```

Function Lachesis(\mathcal{T}, n)

in : A semantic tableau \mathcal{T} , and a node n .
output : True or False.

```

1 begin
2   foreach eventuality  $\epsilon \in \mathcal{T}(n)$  do
3     | if  $\epsilon$  is not fulfilled in any path  $\pi = \langle n, \dots \rangle$  then
      | return False;
4   return True;

```

most $O(|B|^{l^2 \cdot (|\phi|+1)^3} \cdot 2^{\text{length}(\phi)})$ nodes.

By Lemma 1 it can be easily inferred that the decision procedure requires time and space exponential in the length of an initial formula. It is however possible to test a \mathcal{L}_1 formula for satisfiability using only polynomial space in the length of that formula. The satisfiability problem for a \mathcal{L}_1 formula is in fact complete in PSPACE if the considered qualitative spatial constraint language has compactness and patchwork for atomic QCNs [Sioutis *et al.*, 2015].

To decide the satisfiability of a \mathcal{L}_1 formula ϕ using the tableau that is generated by Clotho, we have to eliminate unsatisfiable nodes inductively, until a fixed point is reached. We present Atropos, an algorithm that achieves this goal, shown in Algorithm 2. If the root node is eliminated after the application of Atropos, we call the tableau *closed*, and *open* otherwise. Note that function Lachesis essentially searches for a path from a given pre-state to a node that fulfills an eventuality of the pre-state, as defined in Definition 9.

Example. Let us consider formula $\phi = \{EQ(x, y), PO(\circ x, \circ y), TPP(x, \circ x), TPP(y, \circ y), TPP(x, \circ y), \diamond DC(x, y)\}$. (For simplicity we assume that the decomposition rule for \wedge has already been applied and resulted in the current set form for formula ϕ .) The tableau obtained by the application of algorithms Clotho and Atropos for this formula is shown in Figure 3. Horizontal dotted lines distinguish between different points in time, thus, our tableau extends over three points of time. The root node is 1, the states are 5 to 12, 14, 15, 17, and 18, and the pre-states are 1, 13, and 16. By decomposing the initial formula using the tableau rules and populating it with universal relations where appropriate, we reach states 5 to 12, each one of which harbors a set of base relations that correspond to an atomic

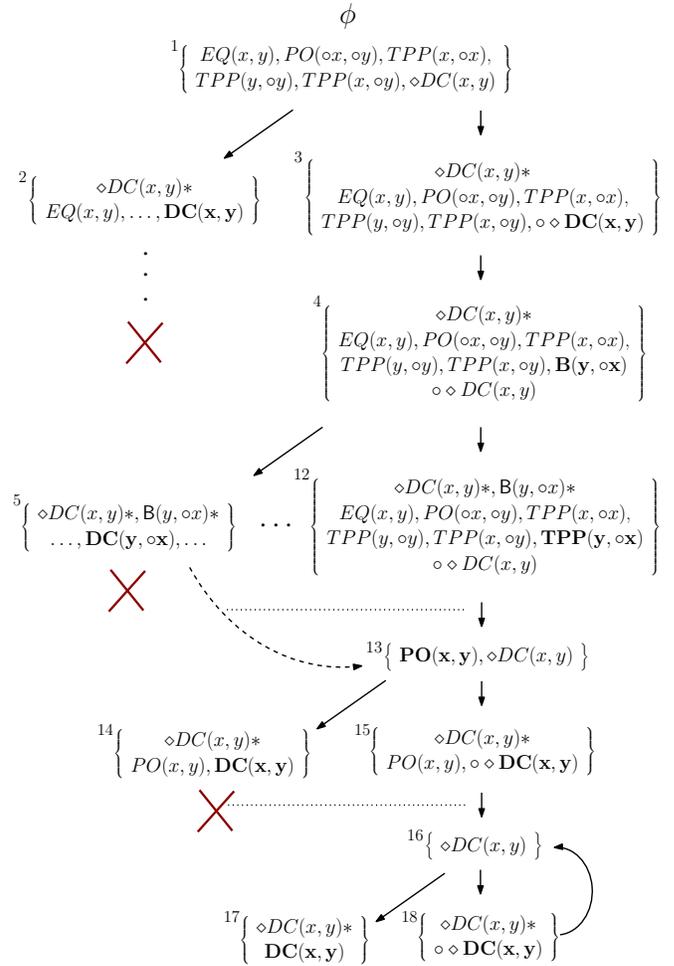


Figure 3: A \mathcal{L}_1 formula and its simplified tableau

QCN. (Inverse relations are not shown to save space.) These atomic QCNs represent a set of atomic spatial constraints in a fixed-width window of time. After leaping to the next point of time and, consequently, obtaining pre-state 13, we include all the atomic spatial constraints of the aforementioned QCNs that extend from the new point of time to a future point of time. In this particular case, the atomic spatial constraints of interest narrow down to the single atomic constraint $PO(\circ x, \circ y)$, common for all states 5 to 12. Of course, since we are now at the next point of time, the constraint is rewritten to $PO(x, y)$. Again, we apply the rules and reach states 14 and 15, each one of which harbors an atomic QCN. We continue repeating the process until all our child nodes are labeled by sets of formulas already met in nodes of the tableau. In this case, the unique child node of state 18 would be labeled by the set of formulas of node 16, thus, we create an arc from 18 to 16. After having constructed our tableau, we delete unsatisfiable nodes 2, 5 to 11, and 14 using the \diamond -consistency operation on QCNs $\mathcal{N}(2)$, $\mathcal{N}(5)$ to $\mathcal{N}(11)$, and $\mathcal{N}(14)$ respectively. Inconsistencies stemming from nodes 5 to 11 stem from the fact that

relation $TPP(y, \circ x)$ is inferred by \diamond -consistency, which contradicts with the base relation that is defined on variables y and $\circ x$ in states 5 to 11. Formula ϕ is satisfiable, as the tableau is open, and a model can be constructed out of the sequence of states 12,15,17 which contains a self loop on 17 as relation $DC(x, y)$ repeats itself. These states harbor satisfiable atomic QCNs that completely agree on their common part due to our construction. In particular, we have the sequence $\mathcal{N}(12) \rightarrow \mathcal{N}(15) \rightarrow \mathcal{N}(17) \circ$ that satisfies the prerequisites of Proposition 2, hence, satisfiability is met.

4.3 Soundness and completeness of our semantic tableau method

We prove that the tableau method as defined by algorithms Clotho and Atropos is sound and complete for checking the satisfiability of a \mathcal{L}_1 formula ϕ that is defined on a language having compactness and patchwork for atomic QCNs.

Theorem 1 (soundness) *If ϕ has a closed tableau, then ϕ is unsatisfiable.*

Proof. Let \mathcal{T} be a closed tableau for ϕ , that has resulted after the application of algorithms Clotho and Atropos. We prove by induction that if a node n is eliminated, then $\mathcal{T}(n)$ is an unsatisfiable set of formulas. We distinguish three scenarios:

- (i) a node n is eliminated because $\mathcal{N}(n)$ is an unsatisfiable QCN (lines 4 to 5 in Atropos), thus, $\mathcal{T}(n)$ is an unsatisfiable set of formulas; unsatisfiability of $\mathcal{N}(n)$ can be detected by use of \diamond -consistency, which also disallows the conjunction of two or more base relations to be defined on a same pair of variables (base relations are jointly exhaustive and pairwise disjoint as noted in Section 2).
- (ii) a node n is eliminated because all of its child nodes are unsatisfiable and have been eliminated (lines 6 to 7 in Atropos). Child nodes can be created in the following three cases:
 - (a) the decomposition rule $\psi \rightarrow \Gamma$, where $\psi \in \mathcal{T}(n)$, is applied and a child node is created for each $\gamma \in \Gamma$ (lines 4 to 9 in Clotho); we have that ψ is satisfiable iff $\exists \gamma \in \Gamma$ that is satisfiable.
 - (b) implicit knowledge in the parent node n is made explicit in the child node m through the introduction of the universal relation B (lines 10 to 19 in Clotho); by Definition 7 we have that $\mathcal{N}(n) = \mathcal{N}(m)$, thus, $\mathcal{N}(n)$ is satisfiable iff $\mathcal{N}(m)$ is satisfiable, and the same holds for the set of formulas $\mathcal{T}(n)$ and $\mathcal{T}(m)$.
 - (c) node n is a state and generates pre-state m with $\mathcal{T}(m) = \{\psi \mid \circ \psi \in \mathcal{T}(n)\} \cup \{P(\circ^{i-1}u, \circ^{j-1}v) \mid P(\circ^i u, \circ^j v) \in \mathcal{T}(n) \text{ if } i, j \geq 1\}$ (lines 20 to 24 in Clotho); clearly, $\mathcal{T}(n)$ is a satisfiable set of formulas iff $\{\psi \mid \circ \psi \in \mathcal{T}(n)\}$ is a satisfiable set of formulas and iff $\mathcal{N}(m)$ is satisfiable.
- (iii) a node n is eliminated if it contains an eventuality that cannot be fulfilled in any path in the tableau (lines 8 to 9 in Atropos); since any model will correspond to a path in the tableau, we have that $\mathcal{T}(n)$ is an unsatisfiable set of formulas.

This concludes our proof. \dashv

Let us obtain a proposition that denotes that two successive states in a path of an open tableau harbor QCNs that completely agree on their common part.

Proposition 3 *Let π be a path going through an open tableau \mathcal{T} for a \mathcal{L}_1 formula ϕ that has resulted after the application of algorithms Clotho and Atropos, s_t and s_{t+1} two states of π belonging to points of time t and $t + 1$ respectively, and $\{x_0, \dots, x_l\}$ the set of variables in ϕ . Then we have that $\mathcal{N}(s_t)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{t+1})[v_m^{k-1}, v_{m'}^{k'-1}] \forall m, m' \in \{0, \dots, l\}$ and $\forall k, k' \in \{1, \dots, |\phi|\}$.*

Proof. State s_t at point of time t is followed by a pre-state p at point of time $t + 1$ in path π , whose set of base relations is $\{P(\circ^{i-1}u, \circ^{j-1}v) \mid P(\circ^i u, \circ^j v) \in \mathcal{T}(s_t) \text{ if } i, j \geq 1\} \cup \{P(\circ^i u, \circ^j v) \mid \circ P(\circ^i u, \circ^j v) \in \mathcal{T}(s_t)\}$ by construction of our tableau (lines 20 to 24 in Clotho). The set of base relations of $\mathcal{T}(p)$ is carried over, possibly enriched, to state s_{t+1} at point of time $t + 1$. As such, let us assume that there exists an additional base relation $b(\circ^{i-1}u, \circ^{j-1}v)$ in the set of base relations of s_{t+1} , with $i, j \in \{1, \dots, |\phi|\}$, such that $b(\circ^i u, \circ^j v) \notin \mathcal{T}(s_t)$. In this case, $\mathcal{N}(s_{t+1})$ is a QCN with two base relations defined on a same pair of variables. This QCN would have been deleted by the application of Atropos as specified also in the proof of Theorem 1. Thus, state s_{t+1} could not have been in path π , resulting in a contradiction. Therefore, we have that $\mathcal{N}(s_t)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{t+1})[v_m^{k-1}, v_{m'}^{k'-1}] \forall m, m' \in \{0, \dots, l\}$ and $\forall k, k' \in \{1, \dots, |\phi|\}$, and, as such, $\mathcal{N}(s_t)$ and $\mathcal{N}(s_{t+1})$ completely agree on their common part. \dashv

Theorem 2 (completeness) *If ϕ has an open tableau, then ϕ is satisfiable.*

Proof. Let \mathcal{T} be an open tableau for ϕ , that has resulted after the application of algorithms Clotho and Atropos. We need to show that there exists a path of nodes π which defines a model for ϕ . We distinguish two scenarios:

- (i) if no eventualities need to be fulfilled, path π can be simply a path starting from the root node and going through the tableau, defining a sequence of states s_0, s_1, \dots, s_t , with $t \in \mathbb{N}$, and, consequently, yielding a sequence of QCNs as follows:

$$\mathcal{N}(s_0) \rightarrow \mathcal{N}(s_1) \dots \rightarrow \mathcal{N}(s_t)$$

The sequence of QCNs is such that for all states s_i and s_{i+1} , with $i \in \{0, \dots, t-1\}$, along with a set of variables $\{x_0, \dots, x_l\}$ in ϕ , we have that $\mathcal{N}(s_i)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{i+1})[v_m^{k-1}, v_{m'}^{k'-1}] \forall m, m' \in \{0, \dots, l\}$ and $\forall k, k' \in \{1, \dots, |\phi|\}$ by Proposition 3. Thus, the sequence of QCNs corresponds to the sequence shown in Figure 1, satisfies the prerequisites of Proposition 1, and is therefore satisfiable.

- (ii) if eventualities need to be fulfilled, we show how we can construct a path π that fulfills all eventualities as follows. For each pre-state $p \in \mathcal{T}$ containing an eventuality, we must find a path $\pi_p = \langle p, \dots \rangle$ starting from p , such that all the eventualities contained in p are fulfilled in π_p . We fulfill all the eventualities of p , one by

one, as follows. For a selected eventuality $\epsilon \in \mathcal{T}(p)$, it is possible to find a path $\pi_p = \langle p, \dots, p' \rangle$ in which ϵ is fulfilled and whose last node is a pre-state p' , as otherwise the node would have been deleted by the application of Atropos. By construction of our tableau, p' will also contain the rest of the eventualities that need to be fulfilled (they are carried over from p to p'), and it follows that we can extend path π_p to fulfill a second one, and so on, until all the eventualities of p are fulfilled. By linking together all paths $\pi_p \forall$ pre-states $p \in \mathcal{T}$, we can obtain a path π starting from the initial node and going through the tableau, defining a sequence of states s_0, s_1, \dots, s_{t-1} , with $t \in \mathbb{N}$, with a final loop between state s_{t-1} and a state $s_{t'}$, with $0 \leq t' \leq t - 1$. The loop exists due to the fact that at point of time $t - 1$ there exists a node n , whose child node m is such that $\mathcal{T}(m) = \mathcal{T}(o)$, where o is a node at point of time t' . In particular, we can view a sequence of QCNs as follows:

$$\mathcal{N}(s_0) \rightarrow \mathcal{N}(s_1) \dots \rightarrow \mathcal{N}(s_{t'}) \dots \rightarrow \mathcal{N}(s_{t-1})$$

The sequence of QCNs is such that for all states s_i and s_{i+1} , with $i \in \{0, \dots, t - 2\}$, along with a set of variables $\{x_0, \dots, x_l\}$ in ϕ , we have that $\mathcal{N}(s_i)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{i+1})[v_m^{k-1}, v_{m'}^{k'-1}] \forall m, m' \in \{0, \dots, l\}$ and $\forall k, k' \in \{1, \dots, |\phi|\}$ by Proposition 3. Further, if we were to extend path π , we would obtain a state s_t with $\mathcal{N}(s_t)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{t'})[v_m^k, v_{m'}^{k'}] \forall m, m' \in \{0, \dots, l\}$ and $\forall k, k' \in \{0, \dots, |\phi|\}$ (i.e., s_t replicates the same set of spatial constraints with $s_{t'}$, hence, the loop). Thus, the sequence of QCNs corresponds to the sequence shown in Figure 2, satisfies the prerequisites of Proposition 2, and is therefore satisfiable.

This concludes our proof. \dashv

5 Conclusion and future work

In this paper, we considered a generalized qualitative spatiotemporal formalism, namely, the \mathcal{L}_1 logic, which is the product of the combination of PTL with any qualitative spatial constraint language, such as RCC-8, Cardinal Direction Algebra, and Block Algebra, and presented a first semantic tableau method, that given a \mathcal{L}_1 formula ϕ systematically searches for a model for ϕ . The method presented builds on the tableau method for PTL of Wolper, and makes use of the constraint properties of patchwork and compactness to ensure soundness and completeness. An interesting future direction would be to consider domain interpretations that involve determined entities (constants) for a given qualitative spatial constraint language, and not just abstract infinite domains as it is normally the case.

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